Deep Inference

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Deep Inference

Outline

1 (2 slides) One-sided sequent calculus and Schütte's 'deep' cut rule

2 (9 slides) Presentation of systems:

deep inference and premise / conclusion symmetry

→ atomicity and locality:

- Switch rule
- Atomic cut rule
- Promotion rule (and atomic cut)
- Medial rule and atomic contraction
- Seq rule and BV
- Exchange rule of cyclic linear logic
- Classical logic
- Linear Logic
- Conclusions

Outline

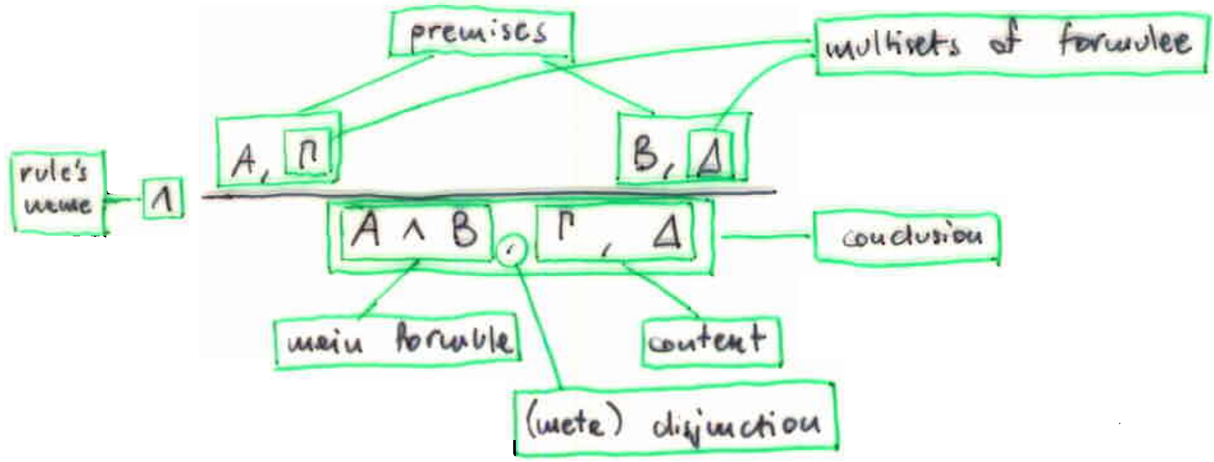
3 (4 slides) Structural properties of systems:

- Decomposition
- Splitting
- Cut elimination
- Conclusions

1 One-sided sequent calculus and Schütte's cut rule

Gentzen-Schütte sequents (GENTZEN, PRAWITZ, SCHÜTTE)

- Example: 'and' rule for classical logic, with multiplicative context treatment:



- Interpretation: if ((A or Γ) and (B or Δ) are provable then (A∧B) or Γ or Δ are provable.
- interpretation: $((A \vee \Gamma) \wedge (B \vee \Delta)) \rightarrow ((A \wedge B) \vee \Gamma \vee \Delta)$; this doesn't always hold of the sequent calculus but it holds for us.
- This rule is shallow because the main formula is at the top level in the context.
- Philosophy: the meaning of '∧' is (apparently) independently defined (Prawitz).

Shallow and Schütte's deep cut rules (GENTZEN, SCHÜTTE)

- Shallow cut rule:

dual formulae

$$\frac{\boxed{A}, \Gamma \qquad \boxed{\bar{A}}, \Delta}{\Gamma, \Delta}\ \text{cut}$$

  - Remark: this rule is not local nor atomic, since A can be any formula → cost of implementing it is unbounded.

  - Remark: cut is sometimes admissible (i.e., unnecessary for completeness) → an atomic cut can be used (at the cost of a complex cut elimination).

- Schütte's cut rule:

$S\{\ \}$ is a deep context

$$\frac{\boxed{S}\{A\} \qquad A \to B}{\boxed{S}\{B\}}\ \text{d-cut}$$

  - $S\{\ \}$ is a formula with a hole $\{\ \}$ at any depth.

  - This rule has no premise/conclusion symmetry → Schütte's 'deep' proof theory does not essentially differ from the classical one.

Switch rule

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$ (this rule is called switch)

- $S(R,T)$ stands for $S\{(R,T)\}$.
- $(R,T) = (T,R)$ stands for (R 'conjunction' T).
- $[R,T] = [T,R]$ stands for (R 'disjunction' T).
- In classical logic switch corresponds to

$$((R \vee U) \wedge T) \to ((R \wedge T) \vee U).$$

- In linear logic switch corresponds to

$$((R \parr U) \otimes T) \multimap ((R \otimes T) \parr U).$$

- Switch uses deep inference.
- Switch is local: its computational cost is bounded.
- Formulae modulo equations are called structures.
- The new formalism I'm dealing with is called calculus of structures.

Atomic cut rule

atoms

$$\mathsf{ai}\uparrow \frac{S(\boxed{a},\boxed{\bar a})}{S\{\boxed{\bot}\}} \qquad \text{(this rule is called atomic cut)}$$

unit of disjunction

- Generic cut rule:

$$\mathsf{i}\uparrow \frac{S(R,\bar R)}{S\{\bot\}} \,.$$

- Easy reduction of i↑ to ai↑:
  - We assume to have De Morgan laws; negation is pushed to atoms.
  - Reduction:

$$\mathsf{i}\uparrow \frac{S((R,T),[\bar R,\bar T])}{S\{\bot\}} \quad \leadsto \quad \text{deep!} \rightarrow \; \mathsf{s}\,\dfrac{S(R,(T,[\bar R,\bar T])) = S((R,T),[\bar R,\bar T])^{*}}{\mathsf{i}\uparrow\,\dfrac{S(R,[\bar R,(T,\bar T)])}{\mathsf{i}\uparrow\,\dfrac{S(R,\bar R)}{S\{\bot\}} = S(R,[\bar R,\bot])}} \,.$$

* structures are associative

- Easy reduction of shallow cut to i↑:

$$\frac{([A,\Gamma],[\bar A,\Delta])}{[\Gamma,\Delta]} \quad \text{corresponds to} \quad \mathsf{s}\,\dfrac{([A,\Gamma],[\bar A,\Delta])}{\mathsf{s}\,\dfrac{[([A,\Gamma],\bar A),\Delta]}{\mathsf{i}\uparrow\,\dfrac{[(A,\bar A),\Gamma,\Delta]}{[\Gamma,\Delta]}}} \,.$$

Promotion rule (and atomic cut) (STRAßBURGER)

$$p\downarrow \frac{S\{![R,T]\}}{S[!R,?T]}$$ (this rule is called promotion)

- This rule is local, contrary to the same one in the sequent calculus:

$$\frac{R, ?T_1, \ldots, ?T_n}{!R, ?T_1, \ldots, ?T_n} .$$

- Reduction of shallow promotion to $p\downarrow$:
  - $??R = ?R$ (and $!!R = !R$).
  - In linear logic R is provable iff $!R$ is provable.
  - Reduction:

$$p\downarrow \frac{p\downarrow \frac{p\downarrow \frac{![R,?T_1,\ldots,?T_n]}{\vdots}}{[![R,?T_1],?T_2,\ldots,?T_n]}}{[!R,?T_1,\ldots,?T_n]}$$

- Easy reduction of shallow cut to $i\uparrow$:

$$i\downarrow \frac{S(!R,?\bar{R})}{S\{\bot\}} \quad \rightsquigarrow \quad i\uparrow \frac{p\uparrow \frac{S(!R,?\bar{R})}{S\{?(R,\bar{R})\}}}{S\{\bot\}} = S\{?\bot\}$$

← copromotion, the dual of promotion

Medial rule and atomic contraction (BRÜNNLER, TIU)

$$\mathsf{m}\,\frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$ (this rule is called medial)

- In classical logic, this rule corresponds to the tautology

$$((R \wedge U) \vee (T \wedge V)) \rightarrow ((R \vee T) \wedge (U \vee V)).$$

$$\mathsf{ac}{\downarrow}\,\frac{S[a,a]}{S\{a\}}$$ (this rule is called atomic contraction)

Contraction can be reduced to atomic contraction by

$$\mathsf{c}{\downarrow}\,\frac{S[(R,T),(R,T)]}{S(R,T)} \quad \rightsquigarrow \quad \mathsf{c}{\downarrow}\,\frac{\mathsf{c}{\downarrow}\,\frac{\mathsf{m}\,\frac{S[(R,T),(R,T)]}{S([R,R],[T,T])}}{S([R,R],T)}}{S(R,T)}\;.$$

Seq rule and BV (Bruscoli, Tiu)

$$q\downarrow \frac{S\langle [R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]}$$ (this rule is called seq)

- $\langle R;T\rangle \neq \langle T;R\rangle$.
- $\overline{\langle R;T\rangle} = \langle \bar{R};\bar{T}\rangle$.
- $\langle \_;\_\rangle$ corresponds closely to CCS prefixing.
- $BV = \{ai\downarrow, s, q\downarrow\}$ (where $ai\downarrow$ is the dual of $ai\uparrow$).
- BV is provably not expressible in the sequent calculus.
- BV is simple.
- Conjecture: BV is Retoré's pomset logic.

Exchange rule of cyclic linear logic (DI GIANANTONIO)

dual to cut → $i{\downarrow}\dfrac{\mathrm{Top}}{[T;\bar{T}]} = [T;(\mathrm{Top};\bar{T})]$

$$\begin{array}{c}\vdots \quad \leftarrow \text{hypothesys} \\ \mathsf{s}\,\dfrac{[T;([R;T];\bar{T})]}{i{\uparrow}\,\dfrac{[T;[R;(T;\bar{T})]]}{[T;R]}} \end{array} = [T;[R;\bot]]$$

- Here $[R;T] \neq [T;R]$ and $(R;T) \neq (T;R)$
- Deep inference takes care of Yetter's weird cyclic exchange rule:

$$\frac{[R;T]}{[T;R]} \;.$$

Classical logic (Brünnler, TU)

down | up

$$\mathsf{ai}\!\downarrow \frac{S\{t\}}{S[a,\bar a]} \qquad \mathsf{ai}\!\uparrow \frac{S(a,\bar a)}{S\{f\}}$$

interaction

structure

$$\mathsf{s}\ \frac{S([R,U],T)}{S[(R,T),U]}$$

core

non-core

$$\mathsf{m}\ \frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$

$$\mathsf{aw}\!\downarrow \frac{S\{f\}}{S\{a\}} \qquad \mathsf{aw}\!\uparrow \frac{S\{a\}}{S\{t\}}$$

$$\mathsf{ac}\!\downarrow \frac{S[a,a]}{S\{a\}} \qquad \mathsf{ac}\!\uparrow \frac{S\{a\}}{S(a,a)}$$

System SKS

- All rules are local, some are atomic.
- The down fragment is complete.
- The up fragment is admissible.
- These achievements are provably impossible in the sequent calculus.

2 Presentation of systems

Linear Logic (STRAßBURGER)

down | up

$$\mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a,\bar a]} \qquad \mathsf{ai}\!\uparrow \frac{S(a,\bar a)}{S\{\bot\}}$$

interaction

structure

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{d}\!\downarrow \frac{S(\langle [R,U],[T,V]\rangle)}{S[\langle R,T\rangle,\langle U,V\rangle]} \qquad \mathsf{d}\!\uparrow \frac{S(\langle R,U\rangle,(T,V))}{S\langle (R,T),(U,V)\rangle}$$

$$\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{p}\!\uparrow \frac{S(?R,!T)}{S\{?(R,T)\}}$$

core

non-core

$$\mathsf{aw}\!\downarrow \frac{S\{0\}}{S\{a\}} \qquad \mathsf{ac}\!\downarrow \frac{S\langle a,a\rangle}{S\{a\}} \qquad \mathsf{ac}\!\uparrow \frac{S\{a\}}{S\langle a,a\rangle} \qquad \mathsf{aw}\!\uparrow \frac{S\{a\}}{S\{\top\}}$$

$$\mathsf{l_0}\!\downarrow \frac{S\{0\}}{S[0,0]} \qquad \mathsf{l}\!\downarrow \frac{S\langle [R,U],[T,V]\rangle}{S[\langle R,T\rangle,\langle U,V\rangle]} \qquad \mathsf{l}\!\uparrow \frac{S(\langle R,U\rangle,\langle T,V\rangle)}{S\langle (R,T),(U,V)\rangle} \qquad \mathsf{l_0}\!\uparrow \frac{S(\top,\top)}{S\{\top\}}$$

$$\mathsf{k_0}\!\downarrow \frac{S\{0\}}{S(0,0)} \qquad \mathsf{k}\!\downarrow \frac{S\langle (R,U),(T,V)\rangle}{S(\langle R,T\rangle,\langle U,V\rangle)} \qquad \mathsf{k}\!\uparrow \frac{S[\langle R,U\rangle,\langle T,V\rangle]}{S\langle [R,T],[U,V]\rangle} \qquad \mathsf{k_0}\!\uparrow \frac{S[\top,\top]}{S\{\top\}}$$

$$\mathsf{m_0}\!\downarrow \frac{S\{0\}}{S\langle 0,0\rangle} \qquad \mathsf{m}\,\frac{S\langle\langle R,U\rangle,\langle T,V\rangle\rangle}{S\langle\langle R,T\rangle,\langle U,V\rangle\rangle} \qquad \mathsf{m_0}\!\uparrow \frac{S\langle \top,\top\rangle}{S\{\top\}}$$

$$\mathsf{x_0}\!\downarrow \frac{S\{0\}}{S\{?0\}} \qquad \mathsf{x}\!\downarrow \frac{S\langle ?R,?T\rangle}{S\{?\langle R,T\rangle\}} \qquad \mathsf{x}\!\uparrow \frac{S\{!\langle R,T\rangle\}}{S\langle !R,!T\rangle} \qquad \mathsf{x_0}\!\uparrow \frac{S\{!\top\}}{S\{\top\}}$$

$$\mathsf{y_0}\!\downarrow \frac{S\{0\}}{S\{!0\}} \qquad \mathsf{y}\!\downarrow \frac{S\langle !R,!T\rangle}{S\{!\langle R,T\rangle\}} \qquad \mathsf{y}\!\uparrow \frac{S\{?\langle R,T\rangle\}}{S\langle ?R,?T\rangle} \qquad \mathsf{y_0}\!\uparrow \frac{S\{?\top\}}{S\{\top\}}$$

$$\mathsf{z_0}\!\downarrow \frac{S\{\bot\}}{S\{?0\}} \qquad \mathsf{z}\!\downarrow \frac{S[?R,T]}{S\{?\langle R,T\rangle\}} \qquad \mathsf{z}\!\uparrow \frac{S\{!\langle R,T\rangle\}}{S(!R,T)} \qquad \mathsf{z_0}\!\uparrow \frac{S\{!\top\}}{S\{1\}}$$

- All rules are local, some are atomic, and they come from common schemes.
- The down fragment is complete, the up fragment is admissible (and this cannot be done in the sequent calculus).

Conclusions

Deep inference + premise / conclusion symmetry → locality + atomicity + uniformity

- Systems in the calculus of structures:
  - Enjoy all the good properties of those in the sequent calculus (e.g., the subformula property, in essence).
  - They have more properties, because rules have finer granularity.
- It is possible to define simple systems for which no sequent calculus implementation is possible.
- Locality and atomicity are of direct computational interest.

Immediate future:

- Modal logics (the sequent calculus is very bad at them)
- Subatomic logic: unification of several logics, including classical and linear logic, with two (2) rules: switch and medial.

Decomposition (BRÜNNLER, STRAßBURGER)

Example of decomposition theorem:

$$\forall \quad \begin{array}{l} T \\ \| \; \mathsf{SKS} \\ R \end{array} \quad \rightarrow \quad \exists \quad \begin{array}{ll} T & \\ \| \; \{ac\uparrow\} & \text{(cocontraction)} \\ T_1 & \\ \| \; \{aw\uparrow\} & \text{(coweakening)} \\ T_2 & \\ \| \; \{ai\downarrow\} & \text{(identity axioms)} \\ T_3 & \\ \| \; \{s, m\} & \text{(switch and medial)} \\ R_3 & \\ \| \; \{ai\uparrow\} & \text{(cut)} \\ R_2 & \\ \| \; \{aw\downarrow\} & \text{(weakening)} \\ R_1 & \\ \| \; \{ac\downarrow\} & \text{(contraction)} \\ R & \end{array}$$

- Several theorems of this kind hold for classical and linear logic and for BV.
- For linear logic and BV decomposition is constructive (for classical logic this is still a conjecture). Proving these results is (still) very hard.
- These results are provably impossible in the sequent calculus.

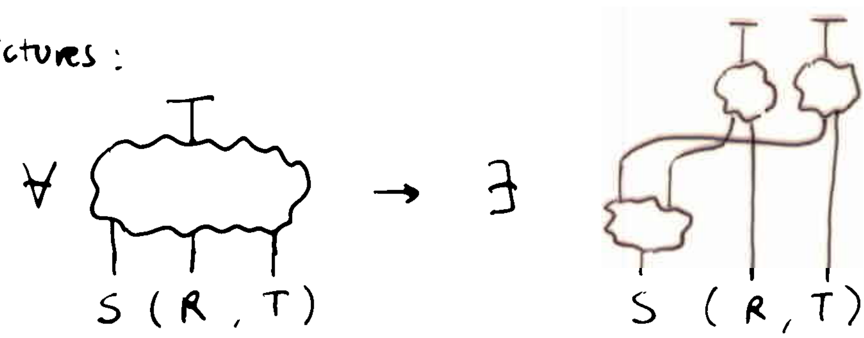
Splitting (STEWART, STRAßBURGER)

Example of splitting theorem:

$$\forall \begin{array}{c} \Pi\,\mathsf{KS} \\ S(R,T) \end{array} \rightarrow \exists \left(\begin{array}{c} [\{\},K_R,K_T] \\ \Vert\,\mathsf{KS} \\ S\{\} \end{array} \text{ and } \begin{array}{c} \Pi\,\mathsf{KS} \\ [R,K_R] \end{array} \text{ and } \begin{array}{c} \Pi\,\mathsf{KS} \\ [T,K_T] \end{array} \right)$$

where KS is the down fragment of SKS and Π_R^{KS} is a proof of R.

In pictures:



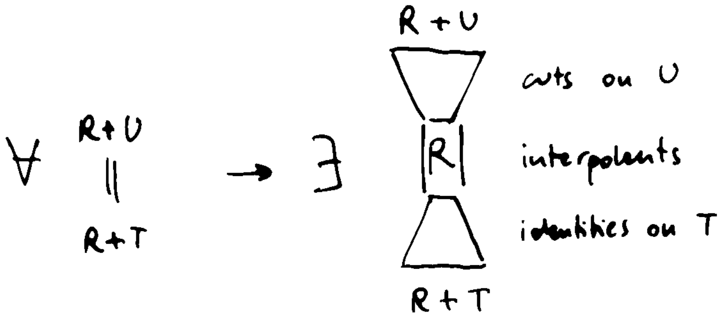
- Splitting theorems hold for all systems in the calculus of structures.
- They're somewhat similar to synchronicity / asynchronicity in linear logic (cfr. ludics).
- They allow nondeterminism control in proof search.
- Proving them is complex but no more than cut elimination.

Cut elimination (BRÜNNLER, DI GIANANTONIO, STEWART, STRAßBURGER, TIU)

Decomposition + splitting $\rightarrow$ cut elimination

- Cut elimination holds for all systems (of course!).
- Cut elimination is just one special case of admissibility: all rules in the up fragment are admissible.
- Generic cut is equivalent to atomic cut + core up.
- Proofs of cut elimination independent of decomposition and splitting exist: SKS admits the simplest syntactical proof of cut elimination ever.

Conjecture:



then cut elimination would be a special case of interpolation for proofs (which is something one cannot do in the sequent calculus).

Conclusions

- While other formalisms have only one notion of normalisation, the calculus of structures admits several (decompositions, admissibility of several rules, interpolation).
- Cut elimination is not central anymore, it's just one phenomenon among several others of a similar nature.
- This could mean a more uniform, deeper proof theory for more logical systems.
- Applications are in sight for the denotational semantics of proofs (so, of computations)

Future developments:

- Proof nets (especially for classical logic).
- Tackling Prawitz's thesis with strong arguments.
- Developing relation web semantics (sort of a cellular automata view of computation), which is where all this started from.