The Calculus of Structures

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1. What Is the calculus of structures?

2. Classical Logic
   atomicity, locality

3. Linear Logic
   modularity

4. System SBV
   calculism (and process algebras)
1. What is the calculus of structures?

It's a step back from the sequent calculus.

New properties:
- atomicity
- locality
- modularity

Do we get a better proof theory?
1. **What is the calculus of structures?**

The sequent calculus is very committed to trees.

**Example 1**  
"Additive" conjunction

\[
\begin{array}{c}
\Gamma, A, B, \Delta \\
\hline
\Gamma, A, B, \Delta \\
\end{array}
\]

The formula tree shapes the proof tree.

**Example 2**  
"Multiplicative" conjunction

\[
\begin{array}{c}
\Gamma, A, B, \Delta \\
\hline
\Gamma, A, B, \Delta \\
\end{array}
\]

The formula tree induces an unwanted tree (in proof-search).
Example 3  Cut elimination

The formula tree decides the order of reductions
1. What is the calculus of structures?

Trees are unfriendly to distributed computation.

- **Example** Suppose that
  - atoms are processors: a, b, c, d
  - communication flows through the tree structure

```
        1
       /\  /
      2 3
     /\  /
    a b c d
```

The communication workload of 1 is four times that of 2 and 3.

- Main connectives create an asymmetry.
- Step back: in the calculus of structures, there are no main connectives.
1. What is the calculus of structures?

There are no main connectives

- Example 1 "Additive" conjunction

\[ \vdash (A \lor C) \land (A \lor C) \]
\[ \vdash (A \land B) \lor C \]

- Example 2 "Multiplicative" conjunction

\[ \vdash (A \otimes C) \oplus B \]
\[ \vdash (A \oplus B) \otimes C \]

- Inference rules can be applied deep inside formulae

- There is a new top-down symmetry

- What happens to the subformulae property?
Example 1  "Additive" conjunction

Rule

\[ p \frac{S \{ (AvC) \land (BvC) \} }{S \{ (AvB) \lor C \} } \]

can be applied as in

\[ p \frac{(AvC) \land (BvC) \land D \lor E}{(AvB) \lor C \land D \lor E} \]

Example 2  "Multiplicative" conjunction

Rule

\[ p \frac{S \{ (AvC) \bullet B \} }{S \{ (AvB) \bullet C \} } \]

can be applied as in

\[ p \frac{(AvC) \bullet (B \bullet D)}{(AvB) \bullet C \bullet D} \]
Inference rules can be applied deep inside formulae.

- Inference rule $p$:

  - The hole in $S \& 3$ does not appear inside a negation.

  - Rule $p$ corresponds to $T \rightarrow R$. 
1. What is the calculus of structures?

Structures

- Atoms are positive or negative: \( a, b, c, \ldots, e, f, g, \ldots \)

- Structures \( P, Q, R, S, T, U, \ldots \) are

\[
S ::= \\
\text{atoms} \quad a \\
\text{disjunctions} \quad \mathbin{\|} (S, \ldots, S) \\
\text{conjunctions} \quad \mathbin{\&} (S, \ldots, S) \\
\text{other relations} \quad \llbracket S, \ldots, S \rrbracket \mid \ldots \\
\text{units} \quad \top, \bot, 1, 0, \ldots \\
\text{modalised structures} \quad \Diamond S, \Box S, \ldots \\
\text{quantified structures} \quad \exists x. S, \forall x. S, \ldots \\
\text{negated structures} \quad \neg S
\]
Structures

- Equations are imposed over structures:

  Commutativity (not always)
  \[ [R, T] = [T, R] \]

  Associativity (always)
  \[ \langle R; \langle T; U \rangle \rangle = \langle R; T; U \rangle \]

  de Morgan (always!)
  \[ \overline{[R, T]} = (\overline{R}, \overline{T}) \]

  contentued closure
  \[ R = T \Rightarrow S\{R\} = S\{T\} \]

- Notation: Braces are dropped when unnecessary.
  Example:
  \[ S[R, T] \text{ instead of } S\{R, T\} \]
If

\[ \frac{S \not R 3}{S \not R 3} \]

is a rule, corresponding to

\[ T \to R \]

then

\[ \frac{S \not R 3}{S \not R 3} \]

is also a rule, corresponding to

\[ R \to \overline{T} \]
What is the calculus of structures?

There is a new top-down symmetry

Example

In linear logic

\[
\begin{align*}
P \Downarrow & \quad S \downarrow !([R,T]) \\
P \Uparrow & \quad S [!R, ?T]
\end{align*}
\]

corresponds to

\[
!(R \& T) \rightarrow (?!R ?T)
\]

corresponds to

\[
P \Uparrow \quad S (?R, !T) \\
S \uparrow (?R, T)^3
\]

corresponds to

\[
(R \& ?T) \rightarrow !R \& T
\]
What is the calculus of structures? 3.16

There is a new top-down symmetry 3 of 3

- Derivations (Δ) are chains of instances of inference rules

  \[ \vdash \]
  \[ \frac{\pi}{T} \]
  \[ \frac{\beta}{R} \]
  \[ \vdash \]

- There is a top-down symmetry. Example

  \[ \vdash \]
  \[ \frac{R}{T} \]
  \[ \frac{\beta}{T} \]
  \[ \frac{\pi}{U} \]
  \[ \vdash \]

is a valid derivation
What is the calculus of structures?

What happens to the subformula property?

- Morally, it still holds if we design rules carefully. Example:
  \[
  \frac{S([R,U], T)}{S([R,T], U)}
  \]

  premise and conclusion are made of the same pieces

- Rules can still be finitary, either upwards, or downwards, or both

- Being finitary does not depend on having main connectives
Do we get a better proof theory?

- We have some chances because:
  - we abolished the main connective idea
  - we are free to apply rules deeply
  - then we have more freedom
  - we also have a new symmetry!
  - we should see proofs in more detail

- But:
  - we have to be careful in designing systems!
    (we shouldn’t abuse freedom)
  - it’s still not clear whether we can do some good distributed computation
Recipe for a good system

- Choose disjunction and conjunction and make identity and cut.

Example: linear logic

- $[R, T]$ stands for $R \& T$
- $(R, T)$ stands for $R \otimes T$

- Establish

\[
\frac{S[R, R]}{S[R, R]} \quad \frac{S(R, R)}{S[R, R]}
\]

- This is your interaction fragment
Take each couple of dual logical relations, for example:

- \( \{R, T\} \) stands for \( R \circ T \)
- \( (R, T) \) stands for \( R \land T \)

and create the rules

\[
\begin{align*}
\frac{S([R,U],[T,V])}{S([R,T],[U,V])} & \quad \frac{S([R,T],[U,V])}{S((R,U),(T,V))} \\
\frac{\exists n.R [\forall n.T]}{S[\forall n.R, \exists n.T]} & \quad \frac{S(\exists n.R, \forall n.T)}{S[\exists n.(R,T)]}
\end{align*}
\]

This is your core structure fragment

Add the non-core structure fragment
A one-sided system into the calculus of structures

One-sided (Gentzen-Schütte) system $GS\vdash p$

A system for classical logic in the calculus of structures (the "nait" system)

\[
\begin{align*}
\text{id} & \quad \frac{\text{id}}{A, \bar{A}} \\
\text{VL} & \quad \frac{\text{VL}}{B, A, \bar{A}} \\
\wedge & \quad \frac{\wedge}{B, A, \bar{A}} \\
\text{V} & \quad \frac{\text{V}}{B, A, \bar{A}} \\
\text{c} & \quad \frac{\text{c}}{B, A, \bar{A}} \\
\text{wt} & \quad \frac{\text{wt}}{B, A, \bar{A}} \\
\end{align*}
\]
2. Classical logic

A one-sided system into the calculus of structures

Equations

\[ [R] = (R) = R \]
\[ [R, \overline{R}] = [\overline{R}, R] \]
\[ (R, \overline{R}) = (\overline{R}, R) \]
\[ \overline{R} = R \]
\[ [R, [\overline{R}, \overline{R}], \overline{R}] = [R, \overline{R}, \overline{R}] \]
\[ (R, (\overline{R}, \overline{R})) = (R, \overline{R}, \overline{R}) \]

\[ [R, \overline{T}] = (\overline{R}, T) \]
\[ (R, T) = [\overline{R}, T] \]

Example: Prove \( ((A \lor B) \lor A) \lor A \equiv ((\overline{A} \lor B) \lor A) \lor A \equiv ((\overline{A} \lor B) \land \overline{A}) \lor A \)

\[
\begin{align*}
\frac{id}{\vdash A, A} \\
\frac{id}{\vdash \overline{A} \lor B, A} \\
\frac{id}{\vdash (\overline{A} \lor B) \land \overline{A}, A} \\
\frac{id}{\vdash ((\overline{A} \lor B) \land \overline{A}) \lor A} \\
\frac{id}{\vdash ((\overline{A} \lor B) \land \overline{A}) \lor A}
\end{align*}
\]
The calculus of structures generalizes the one-sided sequent calculus.

* It is trivial and un-interesting to partake in the one-sided sequent calculus to the calculus of structures.

* The translation works like this:

```
\( \frac{E, \ldots, E_n, E', E''}{E'} \quad \frac{E', E''}{E, \ldots, E_n, E', E''} \)
```

* Symmetry is not exploited!

* Dequeuising is not exploited!

* Can we do better then the sequent calculus?
2 Classical logic

A deep, symmetric system

- Let's apply our recipe!
- We keep the equations we have already

- Interaction

  \[
  \text{lL } \frac{\text{S} \text{tr} \text{3}}{\text{S}[R, R]} \quad \text{rR } \frac{\text{S}[R, R]}{\text{S} \text{tr} \text{3}}
  \]

- Core structure

  \[
  \text{sL } \frac{\text{S}[R, U], [T, V]}{\text{S} [(R, T), U, V]} \quad \text{sR } \frac{\text{S}[R, T], U, V)}{\text{S} [(R, U), (T, V)]}
  \]

- Non-core structure (here we have to be creative)

  \[
  \text{wL } \frac{\text{S} \text{tr} \text{3}}{\text{S} \text{er} \text{3}} \quad \text{wR } \frac{\text{S} \text{er} \text{3}}{\text{S} \text{tr} \text{3}}
  \]

  \[
  \text{cL } \frac{\text{S}[R, R]}{\text{S} \text{er} \text{3}} \quad \text{cR } \frac{\text{S} \text{er} \text{3}}{\text{S}[R, R]}
  \]
A deep, symmetric system

- **Definition** A system $(\mathcal{S})$ is a set of inference rules.

- **Definition** A rule $\phi$ is strongly admissible for a system $\mathcal{S}$ if $\phi \notin \mathcal{S}$ and for every instance $\frac{T}{\mathcal{R}}$, there is a derivation $T \vdash_{\mathcal{R}}$.

- **Definition** This rule is called switch : $\frac{S(\langle r, u \rangle, T)}{S(\langle r, t \rangle, U)}$.

- **Proposition** $s_b$ and $s_t$ are strongly admissible for $s$.

- **Proof**

  $s \frac{s(\langle r, u \rangle, \langle T, v \rangle)}{s(\langle r, t \rangle, \langle T, v \rangle)}$  
  $s \frac{s(\langle r, u \rangle, T, v)}{s(\langle r, t \rangle, U, v)}$  
  $s \frac{s(\langle r, t \rangle, U, v)}{s(\langle r, u \rangle, \langle T, v \rangle)}$

- **Remark** Switch is self-dual.

- **Remark** $s$ is a special case both of $s_b$ and $s_t$. 

2. Classical logic
2. Classical logic

A deep, symmetric system

- We have a system, let’s call it CLC

![Diagram of logical system]

- Is this classical logic? Yes: let’s see

- Remark: $\exists!\#, \#^+, s^3$ (and $\#^6, s^3$) is multiplicative linear logic
A deep, symmetric system

- **Theorem**: Every derivation in GS1 p can be transformed into a derivation in CLC, and if it is cut-free, it remains cut-free.

  **Proof**: CLC is more general than the unit system we saw already. (Just pay attention to contraction in the rule A and notice that)

  $\frac{s, [\gamma, \Delta], [\Delta, \tilde{\Delta}]}{s, [\Delta, ([\gamma, \Delta], \tilde{\Delta})]}$

  $\frac{s, [\Delta, ([\gamma, \Delta], \tilde{\Delta})]}{s, [\gamma, \Delta, (\Delta, \tilde{\Delta})]}$

  $\frac{s, [\gamma, \Delta, (\Delta, \tilde{\Delta})]}{s, [\gamma, \Delta]}$

- Then, CLC is classical logic, because every rule is sound.

- Is there any use for cut and CS?
2. Classical logic

A deep, symmetric system

- What about cut elimination?

- Idea: let's exploit the sequent calculus

- Theorem

Every derivation in CLC can be transformed into a derivation in GSTP

Proof

\[
\frac{\Delta \vdash \text{ind. hyp.}}{\Delta} \quad \frac{\Delta \vdash \text{cut}}{\Delta} \quad \frac{\Delta}{} \quad \Delta \vdash \text{cut} \quad \Delta \vdash \text{cut}
\]

This is easily done for each \( p \)
2. Classical Logic

A deep, symmetric system

- Let's break the symmetry!

- **Definition** A proof is a derivation whose topmost structure is (equivalent to) $\varepsilon$

- **Definition** An inference rule $\phi$ is (weakly) admissible for a system $\mathcal{S}$ if $\phi \in \mathcal{S}$ and for every proof $\frac{\Gamma}{\phi}$, there exists a proof $\frac{\Gamma'}{\phi}$

- **Theorem** $\varepsilon$ is admissible for $\{\text{ib, s, wb, cb3}\}$

  (and there is an algorithmic transformation for it)

**Proof**

\[
\begin{align*}
\Delta & \rightarrow \varepsilon(\Delta) \quad \text{(lots of cuts)} \\
& \downarrow \text{cut elimination} \\
(\text{no cuts}) & \varepsilon'(\Delta') \leftrightarrow \Delta'
\end{align*}
\]
2. Classical logic

A deep, symmetric system

- Do we have a better system than classical logic in the sequent calculus?
  Perhaps, but still ...

- Do we have a better, or interesting, cut elimination procedure?
  Well ...

- Symmetry still is not fully exploited!

- Deepness still is not fully exploited!
Atomicity

- Consider

\[
\text{if } \quad S[e,t] \quad \text{then}\quad S[\{e,t\},\bar{e},\bar{t}]
\]

The it's became "smaller", so they eventually can be replaced by

\[
\text{aid } \quad S[e,t] \quad \text{then}\quad S[\{e,t\},\bar{e},\bar{t}]
\]

This rule is called atomic interaction

- Theorem it is strongly admissible for \{a,b\}

- Nothing unexpected!
Atomicity

- Consider

\[
\begin{align*}
& \quad \frac{S([\bar{R}, \bar{T}], \bar{E}, \bar{P})}{s} \\
& \quad \frac{S([R, \bar{E}], \bar{T}, \bar{P})}{s} \\
& \quad \frac{S([E, \bar{E}], T, \bar{P})}{1} \\
& \quad \frac{S(\bar{R}, \bar{T})}{S[\bar{E}]} \\
& \quad \frac{S(T, \bar{P})}{S[1]}
\end{align*}
\]

The it's, too, become "smaller"; we can replace them by

\[
\begin{align*}
& \quad \frac{S(e, \bar{E})}{S[\bar{E}]} \\
& \quad \frac{S(e, \bar{E})}{S[1]}
\end{align*}
\]

This rule is called atomic cointersection.

- Theorem it is strongly admissible for \( \{2\bar{E}, s\} \)

- This property, due to symmetry, we can exploit!
Atomicity of conteraction (cut)

- Consequences:
  - A simpler cut elimination proof
  - Decomposition theorems

- Curiosities:
  - A different relation between cut, subformula property, and finitarity
  - A simple consistency proof
Classical logic

Finitaryness

- In the sequent calculus, finitaryness (going up) corresponds to the subformula property.

Example

\[ \frac{T \Gamma, A \quad T \Gamma, B}{T \Gamma, A \land B} \quad \text{cut} \quad \frac{T \Gamma, A \quad \text{false \ } \bar{A}}{T \Gamma, A} \]

- Finitary
- A and B are subformulas of \(A \land B\)

- non-finitary
- \(A\) is not necessarily a subformula of the conclusion

- In the calculus of structures, there is no subformula property, but still, all inference rules for classical logic are finitary (going up), except for:

\[
\begin{align*}
\text{up: } & \frac{S\{\epsilon, R\} \quad \epsilon \in \mathbb{S}}{S\{\epsilon\}} \\
\text{and: } & \frac{S\{R, \bar{R}\} \quad R \in \mathbb{S}}{S\{R\}}
\end{align*}
\]

(or \(\text{up: } \frac{S\{R, \bar{R}\} \quad R \in \mathbb{S}}{S\{R\}}\) )
• Rules in the core are always finitary!
  (They just "reshuffle" logical relations)

• Conules in the non-core up fragment are always strongly admissible for their duals, plus switch and intersections:

\[
\begin{align*}
& \frac{S[T]}{S[T]} \\
& \frac{S(T, [R, \overline{R}])}{S[T]} \\
& \frac{S(T, [R, \overline{T}])}{S[T]} \\
& \frac{S[R, (T, T)]}{S[T]}
\end{align*}
\]

• Then the only infinitary rule we are left with is

\[
\begin{align*}
& \frac{S[e, e]}{S[e, e]}
\end{align*}
\]
Finitaryness

- Consider the finitary atomic co-interaction rule:
  \[
  \frac{s(e,\bar{e})}{s} \\
  \text{where } e \text{ or } \bar{e} \text{ appears in } s \leq 3
  \]

- It is easy to eliminate all sit instances that are not taut instances, in proofs.

  \[
  \frac{t}{s} \\
  \frac{s(e,\bar{e})}{s} \\
  \frac{R}{F}
  \]

  replace here all e's with t and all \bar{e}'s with f: the proof remains valid!

  proceed inductively upwards in the proof.

- Theorem: Replacing sit by taut does not affect provability.

- Finitaryness does not morally depend on full-blown cut elimination!
A simple consistency proof

1. Theorem Propositional classical logic is consistent
   Proof We cannot get $\top \models f$ when using falsity.

2. Theorem If $R$ is provable then $\overline{R}$ is not provable
   Proof Suppose we have
   
   \[
   \pi, \top \models R \quad \text{and} \quad \pi, \top \models \overline{R}
   \]
   Then we make $\pi, \top \models t$ and then we flip it:
   
   \[
   [R, \overline{R}] \models f
   \]
   Then we can make
   
   \[
   \vdash t \quad [R, \overline{R}] \models f
   \]
   absurd.
Exploiting deepness

- The following rule is called **mediational**:

\[
\frac{S([R, V], [T, V])}{S([R, T], [V, V])}
\]

- **Medial** is self-dual

- Look at

\[
\begin{align*}
\text{ct} & \quad \frac{S([P, P, Q, Q])}{S([P, P, Q])} \\
\text{ct} & \quad \frac{S([P, P, Q])}{S([P, Q])}
\end{align*}
\]

and

\[
\begin{align*}
\text{ct} & \quad \frac{S([P, Q]), [P, Q])}{S([P, P], [Q, Q])} \\
\text{ct} & \quad \frac{S([P, P], Q)}{S([P, Q])}
\end{align*}
\]

By **mediational**, contractions get "smoother".

- The following rules are called **atomic contraction** and **atomic cocontraction**:

\[
\begin{align*}
\text{ct} & \quad \frac{S[e, e]}{S[e, e]} \\
\text{ct} & \quad \frac{S[e, e]}{S[e, e]}
\end{align*}
\]

- **Theorem**: ct is strongly admissible for βct, μ3, and θct.
Exploiting deepness

- Deepness is essential for getting atomic contraction

- In the sequent calculus, it is impossible to get atomic contraction

- By the way, weakening is easily reduced to atomic form:

\[
\begin{align*}
\text{cut} & \quad \frac{S(p, q)}{S(p, q)} \\
\text{cut} & \quad \frac{S(p, f)}{S(p, f)} \\
\text{cut} & \quad \frac{S(p, q)}{S(p, q)} \\
\text{cut} & \quad \frac{S(p, q)}{S(p, q)}
\end{align*}
\]

... and obviously for consequent
System SKS

This is classical logic
Locality

- Let's call locality the property of a rule requiring bounded effort to be applied.

Example: switch

![Diagram]

- Locality depends on the representation

- Atomicity can be a special form of locality

- There still is much to do for distributed computation (but look at relational fields)

- Applications in complexity?
Why cut elimination is different than in the sequent calculus?

Because in the sequent calculus the main connective "drives" the reduction:

\[
\frac{\Gamma, A \quad \Gamma, B}{\Gamma, \Lambda B} \quad \frac{\Gamma, \Lambda}{\Gamma, \Lambda \Lambda B} \quad \frac{\Gamma, \Delta, \Lambda}{\Gamma, \Delta, \Lambda \Lambda B}
\]

\[
\frac{\Gamma, \Lambda}{\Gamma, \Lambda}
\]
Cut elimination

- In the calculus of structures:

\[
\frac{T}{S}
\]

\[
\frac{S \left( d, (e, b, c, [\bar{c}, \bar{b}, \bar{e}]) \right)}{\left[ d, (e, b, c, [\bar{c}, \bar{b}, \bar{e}]) \right]}
\]

\[
\text{if } \frac{S(R, T, [\bar{R}, \bar{T}])}{\text{S} \{ f \}}
\]

\[
R = (e, b)
\]

\[
T = c
\]

\[
S = [d, b, 3]
\]

What are we supposed to do??

- Freedom has a price

- Atomicity helps a lot!
Theorem \( a \uparrow \) is admissible

Proof

1. Transform cuts into shallow cuts:

\[
\mathrm{cut}_s \quad \frac{[s, (e, \bar{e})]}{s}
\]

2. Permute up super cuts:

\[
\mathrm{cut}_s \quad \frac{(s', s_1)}{[(s'_1, \text{n.a}), (s'_2, \text{n.a})]}
\]

where \( \text{n.a} = \underbrace{(e, \ldots, e)}_{\text{n times}} \)

end \( S' \) is obtained from \( S_1 \) by replacing some \( e \)'s by \( f \);

and \( S'_2 \) is obtained from \( S_2 \) by replacing some \( \bar{e} \)'s by \( f \).
2 Classical Logic

Decompositions

Theorems

- For every $T \models \text{sks}$, there is $R$

- For every $T \models \text{sks}$, there is $R$

- One cannot do these things in the sequent calculus

- We start seeing some modularity
Is there any use for weakening and contraction?

Yes:

- We saw def already for getting and (but that use was trivial)
- In interpolation theorems!

It is always possible to generate derivations such that, if $\vdash T \Rightarrow R$, then

\[
\begin{array}{c}
T \\
\quad \vdash \text{growing}
\end{array}
\quad
\begin{array}{c}
U \leftarrow \text{interpolant} \\
\quad \vdash T \Rightarrow U \Rightarrow R
\end{array}
\quad
\begin{array}{c}
\quad \vdash \text{growing}
\end{array}
\quad
\begin{array}{c}
R
\end{array}
\]
Multiplicative exponential linear logic

System SELS

\[
\begin{align*}
\text{down} & \quad \text{up} \\
\frac{S[e,e]}{S[e,e]} & \quad \frac{S(e,e)}{S[e,e]} \\
S[C(R,T),T] & \quad S[C(R,T),U] \\
p & \quad p_1 \\
\frac{S![R,T]}{S![R,T]} & \quad \frac{S!(R,T)}{S!(R,T)} \\
\omega & \quad \omega_1 \\
\frac{S[L]}{S[R]} & \quad \frac{S[R]}{S[R]} \\
l & \quad l_1 \\
\frac{S[R,R]}{S[R,R]} & \quad \frac{S[R,R]}{S[R,R]} \\
+ \text{ decidable equations, especially } \left\{ \begin{array}{l}
??R = ??R \\
!!!R = !!R
\end{array} \right. 
\end{align*}
\]
Multiplicative exponentiated linear logic

- Interactions are atomic
- Promotion is local!
- Absorption (i.e., contraction) is not atomic

Modularity starts to manifest itself: each of \( \text{ai}, \, \text{pi}, \, \text{ui} \) and \( \text{bi} \) is admissible for the down fragment and can be shown admissible independently (to a certain extent)

So, there are \( 2^4 = 16 \) equivalent systems whose properties are known
**Theorem**

For every $T \vdash_R R$

There is a core of SECS

Proof: Difficult!
• We apply all our techniques and get:

• A system, called SCL5, with 34 rules, 16 of which in the up-down fragment (all admissible, of course): so we have $2^{16} = 65,536$ equivalent systems

• All rules are local (or atomic), including contradictions

• All rules follow our recipe + medial + contraction + weakening, so the system is big but very uniform
### Full Linear Logic

<table>
<thead>
<tr>
<th>Structure</th>
<th>Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S([R,U],T)$</td>
<td>$S([R,T],U)$</td>
</tr>
<tr>
<td>$S([R,U],U,V)$</td>
<td>$S([R,T],U,V)$</td>
</tr>
<tr>
<td>$S([R,T],U,V)$</td>
<td>$S([R,T],(U,V))$</td>
</tr>
<tr>
<td>$S([R,T],(U,V))$</td>
<td>$S([R,T])$</td>
</tr>
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<table>
<thead>
<tr>
<th>Expressions</th>
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<tbody>
<tr>
<td>$S(1)$</td>
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<tr>
<td>$S([a, a])$</td>
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<tr>
<td>$S([0], U)$</td>
</tr>
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<td>$S([R,T], U,V)$</td>
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**System SLLS**
It always holds. How do we prove it?

- MLL: splitting
- NELL: decomposition + splitting
- NALL: splitting
- LL: by translation to the sequent calculus
Idea

- CCS is a language for distributed computation where

\[
a \cdot b | \overline{a \cdot b} \rightarrow 0
\]

- Can we make a logic out of this?

- If so, we want \( \overline{a \cdot b} = a \cdot \overline{b} \)

- Then \( \cdot \) is a non-commutative self-dual logical relation

- Problem: getting this in the sequent calculus is very difficult (let's say impossible, see later)
Recipe!

- Ingredients:
  1. 2 commutative dual logical relations
  2. 1 non-commutative self-dual logical relation
  3. 1 self-dual unit common to all relations

- Recipe:

  Just create an intersection and a core structure fragment (everything is multiplicative, for now)

- We get a very simple system whose proof theory is extremely intricate

- The system is atomic and local
The system

- Rules:

- Equations:

  Commutativity:
  \[
  \begin{align*}
  [\overline{R}, \overline{T}] &= [\overline{T}, \overline{R}] \\
  (R, \overline{T}) &= (\overline{T}, R)
  \end{align*}
  \]

  Associativity:
  \[
  \begin{align*}
  [\overline{\overline{R}}, \overline{T}] &= [\overline{R}, \overline{T}] \\
  (R, (\overline{T})) &= ([R], \overline{T}) \\
  (\overline{R}; (\overline{T}; \overline{v})) &= (\overline{R}; \overline{T}; \overline{v})
  \end{align*}
  \]

  Content clause:
  \[
  \text{if } R = T \text{ then } S[R3 = S3T3]
  \]

  Unit:
  \[
  \begin{align*}
  R &= [R, o] = (R, o) = \langle R; o \rangle = \langle 0; R \rangle \\
  \hat{R} &= R \\
  [\overline{R}, T] &= ([\overline{R}, T]) \\
  (R, T) &= ([R, T]) \\
  \langle R; T \rangle &= \langle R; T \rangle \\
  \hat{o} &= o \\
  \end{align*}
  \]

  Singleton:
  \[
  [R] = (R) = \langle R \rangle = R
  \]
The idea comes from the sequent calculus.
• Definition \( \mathbf{BV} = \frac{1}{2} \mathbf{dib}, s, q \mathbf{b} \)  

• Theorem (Splitting)

- If \( \triangledown \mathbf{Ev} \) then \( [\mathbf{3}, [s_1, s_2]] \)
  
  \( \mathbf{Ev} \) and \( \mathbf{Ev} \)
  
  \( s \mathbf{e} [R, s] \)
  
  \( [T, s_2] \)

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  \( [T, s_2] \)

Proof: Complex, but uniform
System SBV

Cut elimination by splitting

- Theorem $\alpha^T$ is admissible for BV

Proof Splitting

- Theorem $\gamma^T$ is admissible for BV

Proof Splitting

- SBV and BV (and $BV \cup \{a_r^3\}$ and $BV \cup \{a_3^3\}$) are equivalent
Decomposition

Theorem

If $T(R) \subseteq S$ then

Proof Permutations

\[ T \subseteq^3 \]
\[ T \]
\[ \text{core of } S = \{s, q, q^2, q^3\} \]
\[ R \]
\[ \{q, q^3\} \]
Intuitive representation of SBV structures

\[ \langle a; [b, (c, \langle d, e \rangle)] \rangle \]
System SBV

SBV cannot be expressed in the sequent calculus.

$S_1$
SBV cannot be expressed in the sequent calculus.
SBV cannot be expressed in the sequent calculus.

• Theorem \(S_1, S_2, \ldots\) are all provable in SBV if and only if one starts reasoning from the looks.

  **Proof** Use relational fields semantics.

• Theorem There is no system in the (normal) sequent calculus which is equivalent to SBV.

  **Proof** Given any sequent system, produce a structure \(S_k\) whose lock is deeper than the depth of the sequent system.
The calculus of structures

Do we get a better proof theory?
Can we do better than the sequent calculus?

We observe:

- atomicity
- locality

- modularity:
  - in the rules
  - in decompositions
  - in cut elimination arguments

- we easily define logics that 'challenge' the sequent calculus