A CRASH COURSE
in
Kripke Semantics
A Kripke Model is

\[ K = (K, \preceq, D) \]

where \((K, \preceq)\) is a poset

where \(D\) is a \(k\)-indexed family of models \(D = \{D_p : p \in \mathbb{P}\}\)

for each \(p \in \mathbb{P}\)

\[ D_p = (D_p, \mathcal{F}_p, \mathcal{E}_p, \mathcal{R}_p) \]

is a FO \(k\)-model

satisfying

\[ p \equiv q \Rightarrow D_p \preceq D_q \]

\[ c_p = c_q \quad \text{for } c \in \mathcal{C} \]

\[ f_p d = f_p d \quad \text{for } d \in D_p, f \in \mathcal{F} \]

\[ R_p \preceq R_q \quad \text{for } R \in \mathcal{R} \]

(not submodel)

A \(K\)-environment \(\eta : \text{Vars} \rightarrow D\)

\[ \forall p : \text{Vars} \rightarrow D_p \quad \exists \eta_p \quad \eta_p(x) = \eta_{\pi}(x) \]

and for \(p \in \mathbb{P}, A\ an\ atomic\ \phi'(a) \Rightarrow R(b_1, \ldots, b_n)\)

\[ p \models_{\neg \eta} A \iff (\prod b_i \mathcal{F}_p \eta \ldots, \prod b_n \mathcal{F}_p \eta) \Leftrightarrow \mathcal{R}_p \mathcal{F}_p \eta \quad R_p \]

where \(\prod c \mathcal{F}_p \eta = c_p \quad \prod b \mathcal{E}_p \eta = \eta_{\pi}(x)\)

\[ \prod b_1, \ldots, b_n \mathcal{F}_p \eta = f_p (\prod b_1 \mathcal{F}_p \eta \ldots, \prod b_n \mathcal{F}_p \eta) \]
For compound formulas

\[ \Pi \frac{1}{2} A \times B = \Pi \frac{1}{2} A \quad \text{or} \quad \Pi \frac{1}{2} B \]

\[ \Pi \frac{1}{2} A \Rightarrow B \quad \text{for every } g \models p \quad q \vdash A \Rightarrow q \vdash B \]

\[ \Pi \frac{1}{2} \bot \quad \text{never} \]

\[ \Pi \frac{1}{2} \exists x A \quad \text{for some } d \in D \quad \Pi \frac{1}{2} A \quad q \not\models d \]

\[ \Pi \frac{1}{2} \forall x A \quad \text{for all } g \models p \\
\text{all } d \in D \quad \forall \frac{1}{2} A \quad q \not\models d \]

\[ K \frac{1}{2} A \quad \text{means} \quad \Pi \frac{1}{2} A \quad \text{all } p \models k \]

\[ \Gamma \models A \quad \text{means} \quad \text{for each } K \frac{1}{2} y \quad y \models K \frac{1}{2} A \quad \text{then } K \frac{1}{2} A \]

Soundness

\[ \Gamma \models A \quad \Rightarrow \quad \Gamma \models \frac{A}{k, \eta} \]

\[ \text{QF: ind. on } k \text{ of } \eta \text{ of } \Gamma \text{ case } D \]

\[ \Gamma \models A \quad \frac{A, B \models C}{\Gamma, B \models C} \quad \Gamma, A \models B \models C \]

Given any \( p \) define \( k^p = (k^p, \leq^p, D^p) \) as follows

\[ k^p = \{ g : g \leq p \} \quad \leq^p = \leq_p \quad (D^p)_g = D_g \cup q \leq p \]
Lemma

Given \( k \leq \rho \)

\( p \vdash_\frac{1}{2} A \iff k^n \vdash A \)

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Back to soundness

For any \( p, k^n \vdash \Gamma \) so \( k^n \vdash \Lambda \) by IH

\( \Rightarrow p \vdash_\frac{1}{2} A \) hence \( p \vdash_\frac{1}{2} B \) hence \( k^n \vdash B \)

so \( k^n \vdash C \) \(
\vdash \vdash \)

\( \vdash k^n \vdash C \)

\( \square \)

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Complements

\( \Gamma \vdash A \Rightarrow \Gamma \vdash \neg A \)

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Def.

Let \( C \) be a fresh, countable set (or"infinite")

A set \( \Gamma \) of formulas non \( \exists \neg \) \( \exists \) is \( \neg \)-prime if

\[ \Gamma \vdash \exists x \exists y B \Rightarrow \Gamma \vdash \exists x \exists y B \]

\[ \Gamma \vdash \exists x A \Rightarrow \Gamma + A \text{ for some } \exists x \]

Thus def. \( \Gamma \) is a set of \( \neg \)-formulas s.t. \( \Gamma \vdash A \)

A is a \( \neg \)-formula

there is a \( \neg \)-prime extension of \( \Gamma \) (or

s.t. \( \Gamma \vdash A \)

\( k.3 \)
Sketch of Pf

\text{let } \psi \text{ be a num. of disj. set formulas with only many repetitions}

\text{we build } P = P_0 \subseteq P_1 \subseteq \ldots

\hat{P} = \text{U}P

\text{stage } 2n \text{ assume } P_{2n} \text{ built, } \text{P}_{2n} \supset A

\text{by } \begin{cases} \text{P}_{2n} + B_n \lor C_n, & \text{P}_{2n+1} = \text{P}_{2n} \lor \exists B_n \text{ if } \text{P}_{2n} \not\supset A \\ \text{else } \text{P}_{2n} \lor \exists C_n \end{cases}

\text{Claim } \text{P}_{2n} \not\supset A

\text{by else } \begin{cases} \text{P}_{2n}, B_n \lor C_n \not\supset A, & \text{P}_{2n+1} \lor B_n \lor C_n \\ \text{cut } \text{P}_{2n} \not\supset A \\ \text{P}_{2n+1} \not\supset A \end{cases}

\text{stage } 2n+1 \text{ assume } \text{P}_{2n+1} \text{ built, } \text{P}_{2n} \supset A

\text{by } \begin{cases} \text{P}_{2n+1} = \text{P}_{2n} \lor D_n & \text{let } \text{P}_{2n+2} = \text{P}_{2n} \lor D_n \big[ \exists D_n \big] \\ \text{c fresh const in } C \\ \text{ex: show } \text{P}_{2n+2} \not\supset A \text{ using cut.} \end{cases}
Let $\mathfrak{A} = \mathfrak{Cn}$.

If $\mathfrak{A}$ is prime over $\mathfrak{C}$ (ex).

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Now back to completeness.

We build a triple model $K\mathfrak{p}$ s.t.

$K \models \mathfrak{P} \land \mathfrak{P} \not\models \mathfrak{A}$

$K = \{ \mathfrak{P} : \mathfrak{P} \not\models \mathfrak{A} \}$

where $C_0 \subseteq C_1 \subseteq \ldots \subseteq C_n$

is a countable sequence of countable sets

$s/t \ C_0 \cup C_n$ is countably infinite.

$D_{\mathfrak{P}} = \text{closed in } L(\mathfrak{P})$ terms

$\mathfrak{P} \vdash A$ iff $\mathfrak{P} \vdash A$ for atmost $A$ terms.

Then $K \not\models \mathfrak{A}$

$K \models \mathfrak{A}$ (original $A$)
A typed higher-order logic with a formal function calculus:

**Church’s Theory of Types**

(1940)

**Base Types**

\( \text{c} \), \( \text{o} \), \( \ldots \)

**Type Expressions**

\( \text{c}, \text{o}, \ldots \)

**Individuals**

truth-values (propositions)

\( \rho(n) \)

**Modern**

\( \Sigma_{(\text{c.o})} \)

\( \Pi_{(\text{c.o})} \)

\( (\text{c.o}) \)

**Archaic**

\( \Sigma_{(\text{c.o})} \)

\( \Pi_{(\text{c.o})} \)

\( (\forall \exists c) \)

\( (\forall \exists c) \) (for each \( \alpha \))

**Constants, Variables & Terms**

- countably many \( \alpha \)
- \( \Sigma_0, \Pi_0, \ldots, \Sigma_\infty, \Pi_\infty \)
- \( \ldots, \alpha_1, \alpha_2, \ldots \)
A constant of type \( \alpha \)
is a term of type \( \alpha \)

A var. of type \( \alpha \)
is a term of type \( \alpha \)

If \( t \) is a term of type \( \alpha \to \beta \)
and \( u \) is a term of type \( \alpha \)
then \( tu \)
is a term of type \( \beta \)

If \( t \) is a term of type \( \beta \)
then
\[ \lambda x : \alpha . t \]
is a term of type \( \alpha \to \beta \)
- A formula is a term of type.
- An atomic formula is a term of type not \( \lambda \)-equivalent to a term in normal form whose head term is a logical constant.

\[ \exists x \, F_0 \quad \text{abbreviates} \quad \sum_{(x_0 \to 0)} \lambda x. F_0 \]

\( \forall \quad \text{reduction} \quad \rightarrow \quad \lambda \text{-equivalence} \quad \text{reflexive, symmetric, transitive closure of } \beta \)
An Intuitionistic Sequent Calculus

\[ \Gamma \vdash T \quad \Gamma, A \vdash A \quad \Gamma, \bot \vdash \bot \]

\[ \Gamma, B, C \vdash C \quad \Gamma, B \vdash C \]

\[ \Gamma, B, C \vdash C \quad \Gamma, C \vdash C \]

\[ \Gamma, B \vdash B \quad \Gamma, C \vdash C \]

\[ \Gamma, B \vdash B \quad \Gamma, C \vdash C \]

\[ \Gamma, P \vdash C \quad \Gamma, \forall x P \vdash C \]

\[ \Gamma, P \vdash C \quad \Gamma, \exists x P \vdash C \]

\[ \Gamma, \bot \vdash \bot \quad \Gamma, \bot \vdash \bot \]

\[ \Gamma, A \vdash C \quad \Gamma, C \vdash B \quad \text{CUT} \]

\[ \Gamma, A \vdash C \]

\[ \Gamma, A \vdash B \]
**Def:** A Heyting Applicative Structure 

\((D, \text{App}, \text{Const}, \Omega, \omega)\) for ICTT 

is a TAS \((D, \text{App}, \text{Const})\) together with a (definitionally) complete Heyting Algebra \(\Omega\) and a function \(\omega: D_0 \rightarrow \Omega\) such that for each \(f \in D_{a \rightarrow b}\) \((x \in D)\), \(\Omega\) contains the meets \(\wedge_{d \in D_a} \omega(\text{App}(f, d))\) and the following conditions hold:
\begin{align*}
\omega \left( \text{Const} \left( \frac{T_0}{\bot} \right) \right) &= T_a \\
\omega \left( \text{Const} \left( \frac{1}{\bot} \right) \right) &= \bot_a
\end{align*}

\begin{align*}
\omega \left( \text{App} \left( \text{App} \left( \text{Const} \left( \frac{\Lambda_{000}}{\bot} \right), d_1 \right), d_2 \right) \right) &= \\
&= \omega \left( d_1 \right) \Lambda_a \omega \left( d_2 \right)
\end{align*}

\text{better notation}

\begin{align*}
\omega \left( \text{Const} \frac{\Lambda}{d_1 \cdot d_2} \right) &= \omega \left( d_1 \right) \Lambda_a \omega \left( d_2 \right) \\
\omega \left( \text{Const} \frac{\Lambda}{d_1 \cdot d_2} \right) &= \omega \left( d_1 \right) \Lambda_a \omega \left( d_2 \right)
\end{align*}

\begin{align*}
\omega \left( \text{Const} \frac{\Sigma_0(\alpha)}{d} \right) \circ f &= \bigvee_{d \in D_a} \omega \left( f \circ d \right) \\
\Pi &= \bigwedge
\end{align*}
\[ f(x, y) = g(x) \cdot h(y) \]

\[ g(x) = \text{Const}(x) \]

\[ h(y) = p(y)(M(x), y) \]

\[ \text{Type}(D) \]

\[ q(Y) \]

\[ \text{Type}(I_C T) \]

\[ A : \text{Type} \rightarrow D \]

\[ \text{Type}(\text{env}) \]

\[ \text{Type}(D, \text{App}, \text{Const}, \text{e}) \]

\[ \text{Type}(\text{Has-3}) \]
For an F-Hensel system (HAS) of with env $\eta$

- The substitution lemma holds:
  \[ h_\eta (M[N/A]) = h_\eta (M) \]
  \[ h_\eta (x = h_\eta (N)) \]

- $h_\eta$ is sound for $\xi, \xi_\beta, \xi_\eta$

Extend $\Sigma \Delta_\eta$ to a set of premises $\Sigma \Gamma_\eta$, as before, define

\[ \Gamma \vdash A \quad \iff \quad \Gamma \vdash_\eta = [A]_\eta \]

\[ \Gamma \vdash A \quad \iff \quad \text{for all } \alpha, \eta \quad \Gamma \vdash_\eta \alpha \]

\[ \text{THM } \quad \Gamma \vdash_\eta \alpha \quad \iff \quad \Gamma \vdash A \]
For completeness:

\[ \Gamma \models \phi \text{ in all models} \]

there exists a sequent \( \Gamma \) in \( \text{MNSP} \)

\[ \Gamma \vdash \phi \]

Build a "Lindenbaum Algebra" \( \text{HAS} \)

\( \Omega = (D, \circ, \cdot, \leq, \Omega, \omega) \) as follows

- \( D_\omega = \{ \text{nf}(t) : t : \alpha \} \)
- \( t \circ t' = \text{nf}(t,t') \)
- \( [B] = \{ C \mid \Gamma, B \frac{\text{c}t}{\text{c}t} C \text{ or } \Gamma, C \frac{\text{c}t}{\text{c}t} B \} \)
- \( [B] = [C] \text{ iff } \Gamma, B \frac{\text{c}t}{\text{c}t} C \)
- \( \Omega = \{ [B] / B : \alpha \} \)

\( \Omega \) is a DC-HA with \( \top = [\top] \)

\( [B] \cup [C] = [B \circ C] \)
To show $\Omega$ is definitionally complete we must show

$$V, \forall \omega ((\forall x. B\circ d) \rightarrow \alpha D_{x} \Omega \beta)$$

exist and give \([\forall \forall B \in \alpha \forall \beta \beta] \Omega \beta \beta \]

by

$$\Gamma, B[\frac{\alpha}{\gamma}] \rightarrow B[\frac{\beta}{\gamma}]$$

$$\Gamma, \forall \forall B \rightarrow B[\frac{\alpha}{\gamma}]$$

$$\Gamma \rightarrow \forall \forall B = B[\frac{\alpha}{\gamma}]$$

and, similarly:

$$\gamma \Gamma, B[\frac{\alpha}{\gamma}] \rightarrow C \forall \forall \gamma$$

$$\Gamma, \forall \forall B \rightarrow C$$

if

$$\Gamma, C \rightarrow B[\frac{\alpha}{\gamma}]$$

$$\Gamma, C \rightarrow \forall \forall B$$
Thus we have an HRS model with
\[ \gamma(x) = x \]
\[ \text{Const}(c) = c \]
and
\[ \sum_{i} I_{\gamma} \leq \sum_{i} I_{\gamma} \]
\[ \forall I_{\gamma} \]
Aside:
A broader class of models

(for type theories
intuitionistic logics
of existence operators
or partial elements)

[aside, not in lectures]
Local Models, Extent

**Def**

A local HAS \( (\mathcal{L}, \mathcal{H}, \mathcal{S}) \)

\[ D = \langle D, \text{App}, \text{Const}, \omega, \Omega, E \rangle \]

is a

- Heyting Applicative Structure

- An extent function \( E = \{ E_a : a \in \mathcal{A} \} \)

\[ E_a : D_a \rightarrow \Omega \]

satisfying:

\[ \text{Range} \{ E \circ \text{const} \} = \{ T_n \} \]

\[ \omega(\text{App}(\text{Const}(\Sigma_{\text{cong}}), f)) = \bigvee_{d \in D_a} E_a(a) \circ \text{App}(f, d) \]

\[ \omega(T_{\text{cong}} f) = \bigwedge_{\text{App}(\text{Const}(T), f)} E_a(a) \rightarrow \omega(fo_d) \]

and where \( \Omega \) (the Heyting Algebra)

is closed under the following meets \& joins (locally parametrically complete)

\[ \bigwedge_{d \in D_a} E(d) \rightarrow \omega(fo_d) \]

\[ \bigvee_{d \in D_a} E(d) \land \omega(fo_d) \]
**Def:** given an lMHS $D$

a local $D$-assignment $\varphi$

is

- a type-preserving map: $\text{Vars} \to D$
- a $D$-assignment $\varphi: \text{Vars} \to D$

must also satisfy

$$\text{E}[[t]]_{\varphi} = T_o$$

for each term $t$ in lCIT

(where $\text{E}[[t]]_{\varphi}$ is the induced interp.)

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**Local model** $(D, \varphi)$

$D$ a local MHS, $\varphi$ a local $D$-assign.

**Semi-local model** $(D, \varphi)$

$D$ a local MHS, $\varphi$ a global $D$-assign.
Lemma

If $\Omega$ is a [complete] [param. complete]
MAS and $m \in \Omega$

then the $a$ relativized algebra $\Omega^a$
given by $\|\Omega^a\| = \{ v \in \Omega : v \in \Omega_a \}$

$\mu \leq a \mu := \nu = \lambda \mu$

$T_a^\mu := a$

$\nu \rightarrow a \mu := (\nu \rightarrow \lambda \mu) \land a \mu$

$\bot_a^\mu := \bot a$

$\land, \lor, \land, \lor, \forall$ restrictions of $\land, \lor, \land, \lor, \forall$ to $\Omega^a$

is a [complete] [param. complete] HA

and $i : \Omega \rightarrow \Omega^a$ given by $i[v] = v \land a \mu$

is a surjective HA-morphism

(Let $\Omega^a = \Omega / \approx a$)

Lemma

Suppose $D = \langle D, 0, \text{Cont}, w, \Omega, E \rangle$
is an LHAS, $\Omega$ a global env for $D$, $u \in \Omega$

let $D^u = \langle D, 0, \text{Cont}, w, \Omega, E \rangle$ be given by $w^u(d) = w(d) \land a \mu$, $E^u(d) = E(d) \land a \mu$

as above, all other values unchanged.

Then $D^u$ is an LHAS, $\Omega$ decl has $Ed = u$
then $\Psi[x = a]$ is a global $D^u$-env.
ALL COMPLETENESS THUS SEEN SO FAR

REQUIRED Cut.

If we can prove completeness without Cut, we have a semantic ref. of Cut-elimination.

IDEA (familiar from semantic tableaux)

Hinikka sets $\rightarrow$ Partial Valuations

Partial Valuations $\rightarrow$ Models

"candidates"

Ref. Andrews: Resolution in Type Theory
Def: Let \( S \) be a poset with least element \( 0 \) and \( S \) a set (disjoint) of elements decorated with types.

- An \((S,k)\) signed forcing formula \( \text{sf} \) is an expression of the form

\[
B \ p \ H \ A_0
\]

where \( B \in \{E,T,F,\} \), \( p \in \text{f} \), and \( A_0 \) a term of type \( o \) in \( \text{ICTT}(S) \).

Let \( \mathcal{K} \) be a set of \((S,k)\).sf.f.c.s.

- \( p \in \text{f} \) is said to occur in \( \mathcal{K} \) if it occurs in some sf.f in \( \mathcal{K} \).

- \( c \in S \) is a \( p \)-constant in \( \mathcal{K} \) if it occurs in a formula \( A_0 \) in some sf.f \( B \neg H \neg A_0 \) with \( \neg p \).

- A term of \( \text{ICTT}(S) \) is a \( p \)-formula (or term) if all of its constants are \( p \)-constants.
Def: A set $\mathcal{H}$ of $(S, \Lambda)$-sffs is an $(S, \Lambda)$-Huntikka set if it satisfies the following conditions:

1. If $Bp \vdash A \in \mathcal{H}$ then $Bp \vdash pA \in \mathcal{H}$
2. If $Tp \vdash A \in \mathcal{H}$ then $Tp \vdash Bc \in \mathcal{H}$ and $Tp \vdash cB \in \mathcal{H}$
3. If $Fp \vdash ABC \in \mathcal{H}$ then $Fp \vdash Bc \in \mathcal{H}$ or $Fp \vdash cB \in \mathcal{H}$
4. If $Tp \vdash VBC \in \mathcal{H}$ then $Tp \vdash Bc \in \mathcal{H}$ or $Tp \vdash cB \in \mathcal{H}$
5. If $Fp \vdash VBC \in \mathcal{H}$ then $Fp \vdash Bc \in \mathcal{H}$ and $Fp \vdash cB \in \mathcal{H}$
6. If $Tp \vdash ZBC \in \mathcal{H}$ then for every $p', p \in \mathcal{H}$ $Fp \vdash Bc \in \mathcal{H}$ or $Tp \vdash cB \in \mathcal{H}$
7. If $Fp \vdash ZBC \in \mathcal{H}$ then for some $p', p \in \mathcal{H}$ $Tp \vdash Bc \in \mathcal{H}$ and $Fp \vdash cB \in \mathcal{H}$
8. If $Tp \vdash Z0(\alpha)B \in \mathcal{H}$ then for some $\alpha \in \mathcal{S}$ $Tp \vdash cB \in \mathcal{H}$
9. If $Fp \vdash \Xi Bc \in \mathcal{H}$ then for each $p$-formula $c \in \mathcal{H}$ (of type $\alpha$) $Fp \vdash ABC \in \mathcal{H}$
1. If \( Tp \vdash \Pi_0(B) \circ \Phi \) then for each \( \forall \beta \) \( \Phi \) and each \( \forall \)-formula \( C_\alpha \) in \( \Phi \)
\( T_\beta \vdash DC \circ \Phi \)

2. If \( Fp \vdash \Pi_1(B) \circ \Phi \) then for some \( \forall \beta \) \( \Phi \) in \( \Phi \) and some constant \( C_\alpha \) of \( S \)
\( Fp' \vdash DC \circ \Phi \)

Condition for \( Bp \vdash \neg A \equiv Bp \vdash A \supset A \)

**Def.** A Hinikka set is contradictory if one of the following holds:

- for some formula \( A \), some \( p \in \bigcirc \)
  \( Tp \vdash A \) and \( Fp \vdash A \) occur in \( \Phi \)
- for some \( p \)
  \( Tp \vdash \perp \circ \Phi \)
- for some \( p \)
  \( Fp \vdash T \circ \Phi \)

**Def.** Let \( \Gamma \vdash A \) be an \( SFP \) sequence.
A Hinikka set for \( (\Gamma, A) \) is an \( (S, K) \)-Hinikka set s.t.
\[ \{ Tp, A' \vdash Y \} \cup \{ Fp, A \vdash A' \} < K \]

**Def.** An \( (S, K) \)-Hinikka set is closed if all nodes force \( T_0 \), and it is \( K \)-monotone:

- \( Tp \vdash A \circ \Phi \) and \( \forall \beta \) \( \Phi \) occurs in \( \Phi \)
  \( T_\beta \vdash A \circ \Phi \)
If $H$ is a set of $(S, k)$ sffs and $p \in K$

\[ T_p(H) = \exists B : T_q(B \in H), p \vdash \exists B : T_q(B \in H) \]

\[ F_p(H) = \exists B : F_q(B \in H) \]

$H$ is ICTT consistent if

for all $p \in K$

for all $B$ in $F_p(H)$

\[ T_p(H) \vdash B \]

where $\vdash$ means provability in the cut-free ICTT sequent-calculus.
Lemma. For every Ω sequence \( p \mapsto A \) there is a \((K_5)\)-Minihakka set for some partially ordered set \( K \) of least element.

(We build a tree of such)

\( k = \mathbb{N}^+ \) prefix order

\( S = \{S_n : n \in \mathbb{N} \} \) a seq. of countably oo sets.
we simultaneously define the following, for each \( n \in \mathbb{N} \)

- A sequence \( \{ \tau_n : n \in \mathbb{N} \} \) of labelled trees, each path \( \gamma \) of which is an approximation to a Minihakka set, i.e. a set of \( 555 \)'s. \( \{ \tau_n \} = \text{set of paths in } \tau_n \)

- A sequence

\( \{ S^R_{n,m}(p) : n \in \mathbb{N}, \ p \in \mathbb{N}, \ m \in \mathbb{N} \} \)

Each \( S^R_{n,m}(p) \) contains those constants of type \( a \) occurring in some signed formula \( \exists \eta \neg A \) in path \( \eta \) of \( \tau_n \) for some \( q \leq p \)

(Suppression of \( \tau_n,a,n \) or \( p \) means \( \emptyset \) over the missing parameter)

\( S^R_n \) is always monotone in \( n,p \)

At each stage \( n \), we extend some \( \eta \in \tau_n \) to a new paths \( \eta' \in \tau_{n+1} \) or a pair of paths \( \eta_0, \eta_1 \) of \( \tau_{n+1} \)

(Thus \( S^R_n \) is never really updated; rather, e.g.

\( S_{n+1} \supseteq S_n \))
We sometimes treat $S_{n,n}(p)$ as a function of $p$, so adding $ca$ to $S_{n,n}(p)$ is written: add $(p, ca)$ to $S_{n,n}$

Thus

"all consns occurring at or below $p$ at on $n =$ \{ $(p,c) \in S_{n,n} : q \leq p$ \}

• A partition of the formula occurrences at each mode $v$ of $T_n$ into used and unused.

**Stage 0**

$C_0$ is the path $\pi = \{ F \triangleleft H \cdot A ; T \triangleleft H \cdot S_1, ..., T \triangleleft H \cdot T_n \}$

where $\pi_0 = \{ X_2, ..., X_n \}$

**Stage n+1**

Assume $T_n, S_n$ have been constructed, and mode $v$ on path $\pi$ is the least unused mode (depth, left-to-right) - ordering in $T_n$, labelled by $B \cdot p \cdot H \cdot A$.

We now build $T_{n+1}$ by attaching to each path $\pi$ through $v$ the tree described below, and updating $S_{n+1}$ according to the structure of $B \cdot p \cdot H \cdot A$.

After such a step declare $v$ as used occurrence
Cases for $B_{p1A}$

$A$ not in $y_n$.

$S_{n+1} = S_n$

$T_{p1A} B C$

Extend each $\pi$ through $y$ with $< T_{p1A} B, T_{p1A} C >$

$S_{n+1} = S_n$

$E_{p1A} B C$

Extend each $\pi$ through $y$ by $T_{p1B}$ $T_{p1C}$

$S_{n+1} = S_n$

$T_{p1A} Y B C$

$F_{p1A} B C$

Extend each $\pi$ with $< F_{p1A} B, F_{p1A} C >$

$T_{p1A} Y B C$

For each $\pi$ through $y$:

Let $p'$ be least $g = 0$ or $k$ occurring on $\pi$ with $g \geq p$

s.t. neither $F_{g1B}$ nor $T_{g1C}$ occurs on $\pi$

Extend each $\pi$ with $F_{p1A} B$

$H-7$

Attach to the leaf $y$ each $\pi$ through $y$ $B_{p1A} y A$ unless already on $\pi$.
Extend each $\pi$ through $\nu$ as follows:

1. Let $k_0$ be the least natural number such that $p_k$ has not occurred on $\pi$.
2. Let $p' = p_k$ (note it is incomparable to any $q \in [p]$ not $\beta$)

Attach $< TP' H B, FP' H C >$ to $\pi$

$$S_{mH}(p') := \bigcup_{q \in \beta} S_{mH}(q)$$

---

Extend each $\pi$ through $\nu$ as follows:

Let $c_0$ be the first constant in $S_0$ that has not occurred in $\pi$, i.e., $c_0 \notin S_{mH}^{\pi}$. Attach $TP H B c_0$ to $\pi$.

$$S_{mH}^{\pi} := S_{mH}^{\pi} \cup \{c(p, c_0)\}$$

---

Let $c$ be the least I.C.T.(S,$\Sigma_0^{\text{ON}}(\nu)$) formula (in $\gamma, \eta, \xi$) such that $FP H B C$ does not already occur in $\pi$. Attach $< FP H B \Sigma_0^{\text{ON}}(c), FP H B C >$

$$S_{mH} := S_{mH}$$

---

Let $(p, c)$ be the least pair $(p, D)$ with $r \in k$, occurring on $\pi$, $r \neq p$ and with $D$ an I.C.T. term over the language of $S_{mH}(p)$ and $TP H B D$ not already on $\pi$.

Extend $\pi$ with $< TP H B P, TP H B C >$

$$S_{mH} := S_{mH}$$

---

Let $k_0 \in k$ be the least natural number such that $p_k$ does not occur on $\pi$, and $p' = p_k$. Let $c_0$ be the first constant in $S_0$ not on $\pi$. Attach $FP' H B c_0$ to $\pi$.

$$S_{m_H}(p') := \bigcup_{q, \in \beta} S_{m_H}(q) \cup \{c_{a_0}\}$$
\[ T_{p\Pi \approx B} \quad F_{p \Pi \approx B} \text{ treated as } \emptyset \quad p \Pi \equiv B \Rightarrow \]

Now let \( c = U_{\Pi c} \). Every node \( g \in c \) is used.

For each \( \Pi \in c \), let \( K^\Pi = \text{restricting } g, K = (N^k, S) \)
to those \( p \) occurring in \( \Pi \), \( K = \text{set}(\Pi) \).

Each \( K^\Pi \) is easily seen a \( K^\Pi \)-Hunkikka set over \( S^n \).

\[ \square \]

\textbf{Def.} \ A \((S, k)\)-Hunkikka set is \textit{closed} if all nodes.

\[ f \text{ to } f \text{ it is monotone; whenever } \]

\[ T_{p \Pi \in H} \quad \text{and } \quad g \in p \quad \text{occurs in } H \]

\[ T_{g \Pi \in H} \]

\textbf{Note:} Any Hunkikka set can be extended to a closed one.
Then Suppose that

Then there is a consistent $k$-Kumbikka path $π$ for $(Γ, A)$.

Claim: If $π$ is a consistent finite path in some partially developed $G$ for $(Γ, A)$
then at least one of the ways it is extended at some stage when one of its entries is used
preserves consistency.

Suppose $B π + A$ occurs/used at node $v$ on path $π$ in $G$.
We consider all cases

A not in $π - A$. Then path $π$ is extended by a single entry $B π + A$. Say $π$ ends in new path $\overline{π}$ and $B = T$.

Then for some $D$ in $F_p(π)$

Then for some $D$ in $F_p(π)$

$T_p(\overline{π})$, $π_1 + D = T_p(π)$, $π_1 + D$. By the $k$-rule $T_p(π)$, $A + D$
and $π$ is not consistent.

$T_p(π) + A B C$
If the new path is not cons., say $\overline{π}$ is witness $D ∈ F_p(π)$

$T_p(π)$, $π + D$. By $k$, $T_p(π)$, $B A C + D$
so $T_p(π) + B$ contradicts cons. of $π$.

$F_p(π) + A B C$

$F_p(π)$ = $T_p(π)$ if $π$ is cons.

Both are inconsistent, the only pass. is

$T_p(π) + B$, since $B$ only new in $F_p(π)$
$T_p(π) + C$

$⇒ T_p(π) + B A C$, $B A C ∈ F_p(π)$ so $π$ not cons.
If the original path is consistent and another extended path is then \( T_{p\beta}(\alpha), \beta \neq \alpha \) for some \( \beta \) in \( T_{p\beta}(\alpha) \) and \( T_{p\beta}(\alpha) \neq B \). 

But then, by \( \exists a \in T_{p\beta}(\alpha), B \subset C \setminus D \). 

However, \( B \subset C \) is \( T_{p\beta}(\alpha) \) so \( T_{p\beta}(\alpha) \neq D \) & \( \Pi \) is inems. \( \Box \)

Then, \( \exists a \) the extension \( g \Psi \) to \( \Pi \) for some appropriate \( p' \).

If this path is inconsistent and the original one is not, the inconsistency must take place at \( p' \). \( T_{p'\beta}(\alpha) \) inherits all true facts for \( g \geq p \) but \( T_{p'\beta}(\alpha) \) does not inherit (false) facts.

So the only possibility is \( T_{p'(\alpha)} \not\subset T_{p'(\gamma)} \).

i.e. \( T_{p'(\alpha)}, B \neq C \)

whence \( T_{p'(\alpha)} \neq B \subset C \Rightarrow \) inems. \( g \neq \pi \).

\( \Box \) some cases left to reader.
**Lemma**

If a Hintikka path $\mathcal{H}$ (resp. set) is consistent, it is non-contradictory.

If $\mathcal{T}_p \vdash B$ and $\mathcal{F}_p \vdash B$ are both in $\mathcal{H}$, then $\mathcal{T}_p(\mathcal{H}) \vdash B$ so $\mathcal{H}$ is inconsistent.

**Thm**

Let $\mathcal{H}$ be a non-contradictory, closed $(K, S)$-Hintikka set for $(S, A)$.

Then: there is a local $\mathcal{K}$-MS model $D$ (i.e. a Kripke model) for $\mathcal{ICFT}$ agreeing with $\mathcal{H}$, i.e. for every closed $\gamma$-valued formula $B$ in some off $\gamma$, $D \vDash B$ if $\mathcal{T}_p \vdash B \in \mathcal{H}$

\[ \square \]

If $\mathcal{P}$ is a consistent set of logical formulae of $\mathcal{ICFT}$ i.e. for some $D$, $\mathcal{P} \vdash D$.

Then there is a cons. Hintikka set making $\mathcal{P}$ true and $D$ false at the root model of the poset action.

By $\mathcal{H}$, by the preceding then, there is an induced local model $\mathcal{T}_p$ agreeing with $\mathcal{H}$. Hence

Then (Completeness) $\mathcal{P} \vdash A$ in $\mathcal{ICFT}$ iff $\mathcal{T}_p(A) = \top$. 

**Hence**
Core: ICTL admits CUT.

If $\Gamma \vdash A$, $\Gamma, A \vdash B$

then $\Gamma \vdash B$.

4 Suppose $\Gamma \vdash B$. Then in $\mathcal{J}_p$ we have $[A] = T$ by soundness, and also $[B] = T$ since $\mathcal{J}_p$ is a model of $\Gamma, A$. This contradicts the completeness theorem for $\mathcal{J}_p$. □
The details of the completeness theorems, well... some of them.

The local HAS \( D_\alpha = \langle D, \circ, \kappa, \Sigma^2, M, E \rangle \)

\( D_\alpha \) consists of PAIRS \( \langle M, r \rangle \) called \( \mathcal{V} \)-complexes

by Takahashi, Andreev, with

\( M \) an open \( \eta \)-y of type \( \alpha \) over \( \text{ITT}(S) \)

for the appropriate \( S \)

\( r \) a member of the full type hierarchy

over \( D_\alpha, D_\beta \) (defined below)

\( D_\alpha \) will contain all possible truth values (intuitionistic)

consistent with \( \alpha \)

a preliminary def.

\[ H^T_A = \exists p: \text{T}_{\Pi 1}A \text{ occurs in } H_3 \]

\[ H^R_A = \exists p: \forall \exists p \ F_{\Pi 1}A \not\in H_3 \]

[Chad Brown CMU]
$\Omega = \mathcal{O}(k)$ the topological C_alg
if all upwards closed subshifts $t$

$T_2 = k \quad \perp \Omega = \emptyset$
$V_2 = U \quad \Lambda_2 = \emptyset$

$\Rightarrow \Omega = \text{Int}(L^c \cup \Lambda)$

$D_0 = \{ \langle A_0, a \rangle : A_0 \text{ in } \eta \text{-}nf$
occurs in $H$
$H_A \subseteq a \subseteq H_{AP}$ $\exists U$

$\{ \langle A_0, m \rangle : A_0 \text{ in } \eta \text{-}nf$
does not occur in $H$
$\forall \epsilon \Omega$ \}

$D_2 = \{ \langle A_c, c \rangle : A_c \text{ is in } \eta \text{-}nf \}$

$D_{pr} = \{ \langle \eta[m], f \rangle : A \text{ in } \eta \text{-}nf, f : D_0 \to D_0$
and for every $\langle B_0, b \rangle \in D_0$
$\forall \langle B_0, b \rangle \text{ is of the form } \langle \eta[AB], r \rangle$
for some $r$ \}

**Lemma:** for every $A_0$ in $\eta$-nf, there is an $r$
s.t. $\langle A_0, r \rangle \in D_0$

$\rho : T_{pr}(H) \to D_2$
by:

$\begin{align*}
\frac{A \in \Omega}{\rho(A) = \emptyset} & : T_{pr}(H) \subseteq H \text{ } \exists \emptyset \text{ } \\
\frac{A \in \Lambda}{\rho(A) = L} & : \text{ } \exists \text{ } \\
\frac{A \in \Omega}{\rho(A) = L} & : \text{ } \exists \text{ } \\
\frac{\epsilon \in \Lambda}{\rho(\epsilon \circ A) \langle A_0, \rho(B_0, b) : \langle \eta[AB], \rho(\eta[AB]) \rangle \}}$

and on types
1. \( \langle A \chi, a \rangle \cdot \langle B, a \rangle = \langle A \chi B \rangle (a \langle B, a \rangle)^2 \)

2. For each \( a \): \( E_a : D_a \to \mathcal{S}_2 \) is given by

   \[
   E_N = \begin{cases} \varepsilon \varphi \kappa : N \text{ is a term on } S_2 & \text{if } N \text{ occurs in } \mathcal{H} \text{ else } \emptyset \\ \{ \varphi \} & \text{if } \varphi, \varphi' \text{ are closed at levels } p, q \text{ and } q \not\geq p \text{ which are incomparable, then the term } c \text{ exists, but has extent } 0. \end{cases}
   \]

NB: \( E(\langle \bot, \bot \rangle) = T_\alpha \)

\( w : D_0 \to \mathcal{S}_2 \) is given by \( w(a, b) = b \).

\[
\begin{align*}
   k(T_0) &= \langle T_0, T_\alpha \rangle \\
k(D_0) &= \langle D_0, \bot \rangle \\
k(Ca) &= \langle Ca, pc_a \rangle \quad \text{for non-variable nodes} \\
k(\sim) &= \langle \sim, A, \langle D_0, b \rangle, \langle \sim, B, B \rangle \rangle \\
k(\lambda) &= \langle \lambda, A, \langle B, a \rangle, \langle \lambda, B, A, \langle D_0, c \rangle, \langle \lambda, B, B, b \rangle \rangle \rangle \\
k(v) &= \\
k(\exists) &= \langle \exists, A, \langle B, a \rangle, \langle \exists, B, A, \langle D_0, c \rangle, \langle \exists, B, B, b \rangle \rangle \rangle \\
k(\forall) &= \langle \forall, A, \langle M_{\varphi}, A \rangle, \langle \forall, M_{\varphi}, A, \langle D_0, m \rangle, \langle \forall, M_{\varphi}, A, \langle D_0, m \rangle \rangle, \langle \forall, M_{\varphi}, A, \langle D_0, m \rangle \rangle \rangle \rangle \\
k(\exists) &= \langle \exists, A, \langle M_{\varphi}, A \rangle, \langle \exists, M_{\varphi}, A, \langle D_0, m \rangle, \langle \exists, M_{\varphi}, A, \langle D_0, m \rangle \rangle, \langle \exists, M_{\varphi}, A, \langle D_0, m \rangle \rangle \rangle \rangle
\end{align*}
\]

Lemma: \( \rho, k \) well-defined.

Lemma: This defines a local MHS, hence sound \( \square \)