Some definitions for the CUT-ELIM. Thm.

- <u>degree</u> of a formula A, (notation: $|A|$), is the number of logical connectives in A.

- <u>cut-rank</u> of a proof Π in Lk, (notation: $c(\Pi)$):
  - Π axiom, then $c(\Pi)=0$
  - Π has shape $\rho\,\dfrac{\overset{\Pi'}{\Sigma'}}{\Sigma}$: $c(\Pi)=c(\Pi')$
  - Π has shape $\rho\,\dfrac{\overset{\Pi'}{\Sigma'} \quad \overset{\Pi''}{\Sigma''}}{\Sigma}$:
    - if $\rho \neq$ cut then $c(\Pi)=\max\{c(\Pi'), c(\Pi'')\}$
    - if $\rho=$ cut and A is the cut-formula then $c(\Pi)=\max\{c(\Pi'), c(\Pi''), |A|+1\}$

- <u>depth</u> of a proof tree Π (notation $d(\Pi)$):
  - Π axiom, then $d(\Pi)=0$
  - Π has shape $\rho\,\dfrac{\overset{\Pi'}{\Sigma'}}{\Sigma}$ then $d(\Pi)=d(\Pi')+1$
  - Π has shape $\rho\,\dfrac{\overset{\Pi'}{\Sigma'} \qquad \overset{\Pi''}{\Sigma''}}{\Sigma}$ then $d(\Pi)=\max\{d(\Pi'), d(\Pi'')\}+1$.

(- logical depth: similar to depth, but structural rules are not counted).

REDUCTION LEMMA (Tait-Girard)

Given proofs $\dfrac{\Pi_1}{\Gamma \vdash \Delta, mA}$ $\dfrac{\Pi_2}{nA, \Lambda \vdash \Theta}$ in LK^+

where $m, n > 0$ (assume $c(\Pi_1), c(\Pi_2) \leq |A|$)

We can construct a proof in LK^+

$$\dfrac{\Pi}{\Gamma, \Lambda \vdash \Delta, \Theta}$$ such that $c(\Pi) \leq |A|$.

(Moreover $e(\pi) \leq 2(e(\pi_1) + e(\pi_2))$ and if the rules of $\supset$ are omitted then $e(\pi) = e(\pi_1) + e(\pi_2)$.)

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proof by induction on $d(\pi_1) + d(\pi_2)$ where π_1, π_2 are the immediate subtrees of the proof tree

$$\text{cut}' \; \dfrac{\dfrac{\Pi_1}{\Gamma \vdash \Delta, mA} \qquad \dfrac{\Pi_2}{nA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

A cut formula

Many cases, not mutually exclusive one.

Organisation must be careful!

The transformed prove is π.

1st Level of organisation : 4 cases
2nd level : L/R + simple/multiple cut.

(1) The root Π_1 and the root Π_2 are conclusions of some LOGICAL inference, having some occurrences of A, the cut formula, as principal formula. In other words A is ACTIVE in the logical inference.

We must consider two versions:
- simple cut
- cross cuts

(2) Either the root of Π_1 or of Π_2 is the conclusion of some logical rule, or of a cut, or of a structural rule, having $X \neq A$ as principal formula: A is PASSIVE

We will consider the case it happens in the left subtree; on the right it is symmetric.

(3) Either Π_1 or Π_2 is an axiom.

We will consider the left subtree; the right is symmetric

(4) Either the root of Π_1 or of Π_2 is the conclusion of some weakening or contraction on the cut formula A.

- We will consider the left subtree (right is symm)
- Distinguish simple / cross cut.

① A ACTIVE in some logical rule

(i) $\wedge_R$ // $\wedge_L$

$$\text{cut}\ \dfrac{\wedge_R\ \dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, B} \qquad \overset{\pi_2}{\Gamma \vdash \Delta, C}}{\Gamma \vdash \Delta, B \wedge C} \qquad \wedge_{L1}\ \dfrac{\overset{\pi_3}{B, \Lambda \vdash \Theta}}{B \wedge C, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta} \quad \Longrightarrow$$

$$\text{cut}\ \dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, B} \qquad \overset{\pi_3}{B, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

hp: $c(\Pi_1), c(\Pi_2) \leq |A|$

where $A = B \wedge C$

$|A| \geq |B| + 1$ and $c(\pi_1), c(\pi_2) \leq c(\Pi_1)$

$c(\pi_3) \leq c(\Pi_2)$

hence $c(\Pi) \leq |A|$

because $c(\Pi) = \max\{|B|+1, c(\pi_1), c(\pi_2)\}$

<u>cross-cuts</u>: assume $m, n > 0$ (with m or $n = 0$ there are simplifications).

let $A = B \wedge C$

$$\text{cut}'\ \dfrac{\wedge_R\ \dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, mA, B} \qquad \overset{\pi_2}{\Gamma \vdash \Delta, mA, C}}{\Gamma \vdash \Delta, (m+1)A} \qquad \dfrac{\overset{\pi_3}{B, nA, \Lambda \vdash \Theta}}{(n+1)A, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

Obtain Π_1' and Π_2' by applying the inductive hp. to the following proofs:

$$\bowtie'\;\dfrac{\begin{matrix}\pi_1\\ \Gamma\vdash\Delta, mA, B\end{matrix}\qquad \wedge_L\dfrac{\begin{matrix}\pi_3\\ B, nA, \Lambda\vdash\Theta\end{matrix}}{(n+1)A, \Lambda\vdash\Theta}}{\Gamma, \Lambda\vdash\Delta, \Theta, B}\qquad \overset{\text{i.h.}}{\rightsquigarrow}\ \Pi_1'$$

$$\Pi_2'\ \overset{\text{i.h.}}{\leftsquigarrow}\qquad \bowtie'\;\dfrac{\wedge_R\dfrac{\begin{matrix}\pi_1\\ \Gamma\vdash\Delta, mA, B\end{matrix}\qquad\begin{matrix}\pi_2\\ \Gamma\vdash\Delta, mA, C\end{matrix}}{\Gamma\vdash\Delta, (m+1)A}\qquad\begin{matrix}\pi_3\\ B, nA, \Lambda\vdash\Theta\end{matrix}}{B, \Gamma, \Lambda\vdash\Delta, \Theta}$$

Let Π be

$$\text{(str. rules)}\ \dfrac{\bowtie'\;\dfrac{\begin{matrix}\Pi_1'\\ \Gamma, \Lambda\vdash\Delta, \Theta, B\end{matrix}\qquad\begin{matrix}\Pi_2'\\ B, \Gamma, \Lambda\vdash\Delta, \Theta\end{matrix}}{\Gamma, \Gamma, \Lambda, \Lambda\vdash\Delta, \Delta, \Theta, \Theta}}{\Gamma, \Lambda\vdash\Delta, \Theta}$$

They hold:

$$\left.\begin{array}{l} c(\pi_1)\leq c(\Pi_1)\\ c(\pi_2)\leq c(\Pi_1)\\ c(\pi_3)\leq c(\Pi_2)\end{array}\right\}\ \xrightarrow{\text{i.h.}}\ \begin{array}{l} c(\Pi_1'), c(\Pi_2')\leq |A| \text{ and}\\ \text{since } |A|\geq |B|+1\ \ (A=B\wedge C)\\ c(\Pi)=\max(\{|B|+1, c(\Pi_1'),\\ \qquad c(\Pi_2')\})\end{array}$$

✓

(ii) $\vee_R/\vee_L$

$$\mathrm{cut}\ \dfrac{\vee_R\ \dfrac{\overset{\pi_1}{\Gamma\vdash\Delta,B}}{\Gamma\vdash\Delta,B\vee C}\qquad \vee_L\ \dfrac{\overset{\pi_2}{B,\Lambda\vdash\Theta}\qquad \overset{\pi_3}{C,\Lambda\vdash\Theta}}{B\vee C,\Lambda\vdash\Theta}}{\Gamma,\Lambda\vdash\Delta,\Theta} \longrightarrow \mathrm{cut}\ \dfrac{\overset{\pi_1}{\Gamma\vdash\Delta,B}\qquad \overset{\pi_2}{B,\Lambda\vdash\Theta}}{\Gamma,\Lambda\vdash\Delta,\Theta}$$

$c(\Pi_1), c(\Pi_2) \leq |A|$ and $c(\Pi) \leq |A|$ because
$c(\Pi) = \max\{|B|+1, c(\pi_1), c(\pi_2)\}$

and $c(\pi_1) \leq c(\Pi_1)$
$c(\pi_2) \leq c(\Pi_2)$
$c(\pi_3) \leq c(\Pi_2)$

—

cross cut

similar to the previous case (i). Try.

(iii) $\supset_L/\supset_R$: Do it as exercise

(iv) $\neg_R/\neg_R$:

$$\mathrm{cut}\ \dfrac{\neg_R\ \dfrac{\overset{\pi_1}{A,\Gamma\vdash\Delta}}{\Gamma\vdash\Delta,\neg A}\qquad \neg_L\ \dfrac{\overset{\pi_2}{\Lambda\vdash\Theta,A}}{\neg A,\Lambda\vdash\Theta}}{\Gamma,\Lambda\vdash\Delta,\Theta} \longrightarrow \mathrm{cut}\ \dfrac{\overset{\pi_2}{\Lambda\vdash\Theta,A}\qquad \overset{\pi_1}{A,\Gamma\vdash\Delta}}{\Gamma,\Lambda\vdash\Delta,\Theta}$$

$c(\Pi_1), c(\Pi_2) \leq |\neg A| \longrightarrow c(\Pi) \leq |\neg A|$ since

$c(\Pi) = \max\{|A|+1, c(\pi_1), c(\pi_2)\}$ and

$c(\pi_1) \leq c(\Pi_1)$ $\qquad c(\pi_2) \leq c(\Pi_2)$.

cross cuts:

$$
\bowtie\ \dfrac{\neg_R\ \dfrac{\overset{\pi_1}{A, \Gamma \vdash \Delta, m(\neg A)}}{\Gamma \vdash \Delta, (m+1)(\neg A)} \qquad \neg_L\ \dfrac{\overset{\pi_2}{n(\neg A), \Lambda \vdash \Theta, A}}{(n+1)(\neg A), \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}
$$

Take π_1', π_2' obtained by applying the i.h. to, resp.:

$$
\bowtie\ \dfrac{\neg_R\ \dfrac{\overset{\pi_1}{A, \Gamma \vdash \Delta, m(\neg A)}}{\Gamma \vdash \Delta, (m+1)(\neg A)} \qquad \overset{\pi_2}{n(\neg A), \Lambda \vdash \Theta, A}}{\Gamma, \Lambda \vdash \Delta, \Theta, A} \rightsquigarrow \pi_1'
$$

$$
\pi_2' \leftarrow\!\!\rightsquigarrow \quad \bowtie\ \dfrac{\overset{\pi_1}{A, \Gamma \vdash \Delta, m(\neg A)} \qquad \neg_L\ \dfrac{\overset{\pi_2}{n(\neg A), \Lambda \vdash \Theta, A}}{(n+1)(\neg A), \Lambda \vdash \Theta}}{A, \Gamma, \Lambda \vdash \Delta, \Theta}
$$

and take π as:

$$
\bowtie\ \dfrac{\overset{\pi_1'}{\Gamma, \Lambda \vdash \Delta, \Theta, A} \qquad \overset{\pi_2'}{A, \Gamma, \Lambda \vdash \Delta, \Theta}}{\dfrac{\Gamma, \Gamma, \Lambda, \Lambda \vdash \Delta, \Delta, \Theta, \Theta}{\Gamma, \Lambda \vdash \Delta, \Theta}}
$$

$c(\pi_1) \leq c(\Pi_1)$
$c(\pi_2) \leq c(\Pi_2)$
$c(\pi_1'), c(\pi_2') \leq |\neg A|$ and $c(\pi) \leq |\neg A|$ because

$c(\pi) = \max\{|A|+1, c(\pi_1'), c(\pi_2')\}$.

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(v) $\forall R / \forall L$

$$\bowtie\dfrac{\forall_R\dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, B[y/x]}}{\Gamma \vdash \Delta, \forall x.B} \qquad \forall_L\dfrac{\overset{\pi_2}{B[t/x], \Lambda \vdash \Theta}}{\forall x.B, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta} \longrightarrow \bowtie\dfrac{\overset{\pi_1[t/y]}{\Gamma \vdash \Delta, B[t/x]} \qquad \overset{\pi_2}{B[t/x], \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

it might be necessary to rename some eigenvariables in π_1 so that they are distinct from all variables in t.

$c(\Pi_1), c(\Pi_2) \leq |\forall x.B| \qquad c(\Pi) \leq |\forall x.B|$ since

$c(\Pi) = \max\{|B[t/x]| + 1, c(\pi_1[t/y]), c(\pi_2)\}$

$c(\pi_1[t/y]) = c(\pi_1) \leq c(\Pi_1)$

$c(\pi_2) \leq c(\Pi_2)$

cross cut:

$$\bowtie\dfrac{\forall_R\dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, m(\forall x.B), B[y/x]}}{\Gamma \vdash \Delta, (m+1)(\forall x.B)} \qquad \forall_L\dfrac{\overset{\pi_2}{B[t/x], n(\forall x.B), \Lambda \vdash \Theta}}{(n+1)(\forall x.B), \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

let Π_1', Π_2' be obtained by applying the i.h. to these:

$$\bowtie\dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, m(\forall x.B), B[y/x]} \qquad \forall_L\dfrac{\overset{\pi_2}{B[t/x], n(\forall x.B), \Lambda \vdash \Theta}}{(n+1)(\forall x.B), \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, B[y/x]} \rightsquigarrow \Pi_1'$$

and

$$\bowtie\dfrac{\forall_R\dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, m(\forall x.B) B[y/x]}}{\Gamma \vdash \Delta, (m+1)\forall x.B} \qquad \overset{\pi_2}{B[t/x], \Gamma, \Lambda \vdash \Delta, \Theta}}{B[t/x], \Gamma, \Lambda \vdash \Delta, \Theta} \rightsquigarrow \Pi_2'$$

let Π be

$$\mathrm{cut}\ \dfrac{\dfrac{\Pi_1'[t/y]}{\Gamma,\Lambda \vdash \Delta,\Theta,B[t/x]} \qquad \dfrac{\Pi_2'}{B[t/x],\Gamma,\Lambda \vdash \Delta,\Theta}}{\dfrac{\Gamma,\Gamma,\Lambda,\Lambda \vdash \Delta,\Delta,\Theta,\Theta}{\Gamma,\Lambda \vdash \Delta,\Theta}}$$

it might be necessary to rename some eigenvariables in Π_1' to make them distinct from all variables in t.

$$\left.\begin{array}{l} c(\pi_1) \le c(\Pi_1) \\ c(\pi_2) \le c(\Pi_2) \\ c(\Pi_1'),\ c(\Pi_2') \le |\forall x.B| \end{array}\right\} \longrightarrow c(\Pi) \le |\forall x.B|$$

because $c(\Pi) = \max\{|B[t/x]|,\ c(\Pi_1'[t/y]),\ c(\Pi_2')\}$
and $c(\Pi_1'[t/y]) = c(\Pi_1')$.

(vi) $\exists R / \exists L$

$$\mathrm{cut}\ \dfrac{\exists_R\,\dfrac{\dfrac{\pi_1}{\Gamma \vdash \Delta, B[t/x]}}{\Gamma \vdash \Delta, \exists x.B} \qquad \exists_L\,\dfrac{\dfrac{\pi_2}{B[y/x],\Lambda \vdash \Theta}}{\exists x.B,\Lambda \vdash \Theta}}{\Gamma,\Lambda \vdash \Delta,\Theta} \longrightarrow$$

some renaming

$$\mathrm{cut}\ \dfrac{\dfrac{\pi_1}{\Gamma \vdash \Delta, B[t/x]} \qquad \dfrac{\pi_2[t/y]}{B[t/x],\Lambda \vdash \Theta}}{\Gamma,\Lambda \vdash \Delta,\Theta}$$

cross-cut: exercise

② A is passive (on the left); any rule (right) | TG9 2.1

symmetric case if A is on the right subtree, or left logical / str. rules are applied.

(i) $\vee_R$:

$$\bowtie\ \dfrac{\vee_R\ \dfrac{\begin{matrix}\pi_1\\ \Gamma\vdash\Delta, mA, B\end{matrix}}{\Gamma\vdash\Delta, mA, B\vee C} \qquad \begin{matrix}\pi_2\\ nA,\Lambda\vdash\Theta\end{matrix}}{\Gamma,\Lambda\vdash\Delta,\Theta, B\vee C} \quad\longrightarrow\quad \vee_R\ \dfrac{\bowtie\ \dfrac{\begin{matrix}\pi_1\\ \Gamma\vdash\Delta, mA, B\end{matrix} \qquad \begin{matrix}\pi_2\\ nA,\Lambda\vdash\Theta\end{matrix}}{\Gamma,\Lambda\vdash\Delta,\Theta,B}}{\Gamma,\Lambda\vdash\Delta,\Theta,B\vee C}$$

$c(\pi_1)\leq c(\Pi_1)$ $\quad c(\pi_2)\leq c(\Pi_2)$ apply the i.h. on $\Gamma,\Lambda\vdash\Delta,\Theta,B$.

(ii) $\wedge_R$:

$$\bowtie\ \dfrac{\wedge_R\ \dfrac{\begin{matrix}\pi_1\\ \Gamma\vdash\Delta, mA, B\end{matrix} \qquad \begin{matrix}\pi_2\\ \Gamma\vdash\Delta, mA, C\end{matrix}}{\Gamma\vdash\Delta, mA, B\wedge C} \qquad \begin{matrix}\pi_3\\ nA,\Lambda\vdash\Theta\end{matrix}}{\Gamma,\Lambda\vdash\Delta,\Theta,B\wedge C} \quad\longrightarrow$$

$$\wedge_R\ \dfrac{\bowtie\ \dfrac{\begin{matrix}\pi_1\\ \Gamma\vdash\Delta, mA, B\end{matrix} \qquad \begin{matrix}\pi_3\\ nA,\Lambda\vdash\Theta\end{matrix}}{\Gamma,\Lambda\vdash\Delta,\Theta,B} \qquad \bowtie\ \dfrac{\begin{matrix}\pi_2\\ \Gamma\vdash\Delta, mA, C\end{matrix} \qquad \begin{matrix}\pi_3\\ nA,\Lambda\vdash\Theta\end{matrix}}{\Gamma,\Lambda\vdash\Delta,\Theta,C}}{\Gamma,\Lambda\vdash\Delta,\Theta,B\wedge C}$$

$c(\pi_1), c(\pi_2)\leq c(\Pi_1)$
$c(\pi_3)\leq c(\Pi_2)$
Apply the i.h. to the subtrees rooted at $\Gamma,\Lambda\vdash\Delta,\Theta,B$ and $\Gamma,\Lambda\vdash\Delta,\Theta,C$

(iii) $\supset_R$: exercise

(iv) $\neg_R$:

$$\bowtie\,\dfrac{\neg_R\,\dfrac{\overset{\pi_1}{B,\Gamma\vdash\Delta,mA}}{\Gamma\vdash\Delta,mA,\neg B}\qquad \overset{\pi_2}{nA,\Lambda\vdash\Theta}}{\Gamma,\Lambda\vdash\Delta,\Theta,\neg B}\quad\longrightarrow\quad \neg_R\,\dfrac{\bowtie\,\dfrac{\overset{\pi_1}{B,\Gamma\vdash\Delta,mA}\qquad\overset{\pi_2}{nA,\Lambda\vdash\Theta}}{B,\Gamma,\Lambda\vdash\Delta,\Theta}}{\Gamma,\Lambda\vdash\Delta,\Theta,\neg B}$$

$c(\pi_1)\leq c(\Pi_1)$

$c(\pi_2)\leq c(\Pi_2)$; apply the i.h. to $B,\Gamma,\Lambda\vdash\Delta,\Theta$

(v) $\forall_R$:

$$\bowtie\,\dfrac{\forall_R\,\dfrac{\overset{\pi_1}{\Gamma\vdash\Delta,mA,B[y/x]}}{\Gamma\vdash\Delta,mA,\forall x.B}\qquad\overset{\pi_2}{nA,\Lambda\vdash\Theta}}{\Gamma,\Lambda\vdash\Delta,\Theta,\forall x.B}\quad\longrightarrow$$

$$\forall_R\,\dfrac{\bowtie\,\dfrac{\overset{\pi_1[z/x]}{\Gamma\vdash\Delta,mA,B[z/y]}\qquad\overset{\pi_2}{nA,\Lambda\vdash\Theta}}{\Gamma,\Lambda\vdash\Delta,\Theta,B[z/x]}}{\Gamma,\Lambda\vdash\Delta,\Theta,\forall x.B}$$

$c(\pi_1[z/y]) = c(\pi_1)\leq c(\Pi_1)$

$c(\pi_2)\leq c(\Pi_2)$

Apply the i.h. to the subtree rooted $\Gamma,\Lambda\vdash\Delta,\Theta,B[z/x]$.

(vi) $\exists_R$: exercise

(vii) cut:

TG 11
2.3

$$
\bowtie\;\dfrac{\bowtie\;\dfrac{\overset{\pi_1}{\Gamma_1 \vdash \Delta_1, m_1 A, pB} \qquad \overset{\pi_2}{qB, \Lambda_1 \vdash \Theta_1, m_2 A}}{\Gamma \vdash \Delta, mA} \qquad \overset{\pi_3}{mA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}
\longrightarrow
$$

Notation: $m_1 + m_2 = m \qquad \Gamma = \Gamma_1, \Lambda_1 \qquad \Delta = \Delta_1, \Theta_1$

By hyp $c(\Pi_1), c(\Pi_2) \leq |A|$ and

$$c(\Pi_1) = \max\{|B|+1, c(\pi_1), c(\pi_2)\}.$$

$\Rightarrow \; |B| < |A| \qquad c(\pi_i) \leq |A| \qquad i = 1..3$

so, in particular $B \neq A$.

We consider the transformation for $m_1, m_2 > 0$ (special cases are if one between $m_1 / m_2 = 0$).

Π_1' is obtained by applying the transformation to

$$
\bowtie\;\dfrac{\overset{\pi_1}{\Gamma_1 \vdash \Delta_1, pB, m_1 A} \qquad \overset{\pi_3}{mA, \Lambda \vdash \Theta}}{\Gamma_1, \Lambda \vdash \Delta_1, \Theta, pB}
$$

and Π_2' similarly from

$$
\bowtie\;\dfrac{\overset{\pi_2}{qB, \Lambda_1 \vdash \Theta_1, m_2 A} \qquad \overset{\pi_3}{mA, \Lambda \vdash \Theta}}{qB, \Lambda_1, \Lambda \vdash \Theta_1, \Theta}
\;;
$$

then take

$$
\dfrac{\bowtie\;\dfrac{\overset{\Pi_1'}{\Gamma_1, \Lambda \vdash \Delta_1, \Theta, pB} \qquad \overset{\Pi_2'}{qB, \Lambda_1, \Lambda \vdash \Theta_1, \Theta}}{\Gamma, \Lambda, \Lambda \vdash \Delta, \Theta, \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}
$$

$c(\Pi_1') \leq |A|$, $c(\Pi_2') \leq |A|$, $|B| < |A|$

$\Rightarrow$ $c(\Pi) \leq |A|$.

(viii) $>_R$

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, B, B, mA}}{\Gamma \vdash \Delta, B, mA}\,{>_R} \qquad \overset{\pi_2}{mA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, B}\,{\bowtie} \quad \longrightarrow \quad \dfrac{\dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, B, B, mA} \qquad \overset{\pi_2}{mA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, B, B}\,{\bowtie}}{\Gamma, \Lambda \vdash \Delta, \Theta, B}\,{>_R}$$

$c(\pi_1) \leq c(\Pi_1)$
$c(\pi_2) \leq c(\Pi_2)$. Apply the i.h. to the subtree rooted $\Gamma, \Lambda \vdash \Delta, \Theta, B$.

(ix) $<_R$

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, mA}}{\Gamma \vdash \Delta, mA, B}\,{<_R} \qquad \overset{\pi_2}{mA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, B}\,{\bowtie} \quad \longrightarrow \quad \dfrac{\dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, mA} \qquad \overset{\pi_2}{mA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}\,{\bowtie}}{\Gamma, \Lambda \vdash \Delta, \Theta, B}\,{<_R}$$

$c(\pi_1) \leq c(\Pi_1)$
$c(\pi_2) \leq c(\Pi_2)$; apply the i.h. to the subtree rooted $\Gamma, \Lambda \vdash \Delta, \Theta$.

✓

(3) Either Π_1 or Π_2 is an axiom | TG13 3.1
(we consider the left subtree; the other is symmetric)

$$\bowtie\ \dfrac{A \vdash A \qquad \begin{array}{c}\pi_2 \\ \vdots \\ mA, \Lambda \vdash \Theta\end{array}}{A, \Lambda \vdash \Theta} \longrightarrow \quad \succ_R \ \dfrac{\dfrac{\begin{array}{c}\pi_2 \\ \vdots \\ mA, \Lambda \vdash \Theta\end{array}}{}}{A, \Lambda \vdash \Theta}$$

$c(\pi_2) \le c(\Pi_2)$; by hyp $c(\Pi_1), c(\Pi_2) \le |A|$

hence $c(\Pi) \le |A|$.

TG14
4.1

④ The root of π_1 (π_2) is the conclusion of some weakening/contraction on the cut formula A.
(We consider this to happen on the left subtree, π_1).

(i) $<_R$:

$$\dfrac{<_R\dfrac{\begin{matrix}\pi_1\\ \Gamma\vdash\Delta\end{matrix}}{\Gamma\vdash\Delta,A}\qquad\begin{matrix}\pi_2\\ A,\Lambda\vdash\Theta\end{matrix}}{\Gamma,\Lambda\vdash\Delta,\Theta}\bowtie \quad\longrightarrow\quad \dfrac{\begin{matrix}\pi_1\\ \Gamma\vdash\Delta\end{matrix}}{\Gamma,\Lambda\vdash\Delta,\Theta}$$

general case, $m, n > 0$:

$$\dfrac{<_R\dfrac{\begin{matrix}\pi_1\\ \Gamma\vdash\Delta,mA\end{matrix}}{\Gamma\vdash\Delta,(m+1)A}\qquad\begin{matrix}\pi_2\\ nA,\Lambda\vdash\Theta\end{matrix}}{\Gamma,\Lambda\vdash\Delta,\Theta}\bowtie \quad\longrightarrow\quad \bowtie\dfrac{\begin{matrix}\pi_1\\ \Gamma\vdash\Delta,mA\end{matrix}\qquad\begin{matrix}\pi_2\\ nA,\Lambda\vdash\Theta\end{matrix}}{\Gamma,\Lambda\vdash\Delta,\Theta}$$

By hp: $c(\Pi_1), c(\Pi_2) \leq |A| \longrightarrow c(\pi) \leq |A|$ in the first case.
Otherwise $c(\pi_1) \leq c(\Pi_1)$, $c(\pi_2) \leq c(\Pi_2)$ and use the i.h.

(ii) $>_R$:

$$\dfrac{>_R\dfrac{\begin{matrix}\pi_1\\ \Gamma\vdash\Delta,(m-1)A,A,A\end{matrix}}{\Gamma\vdash\Delta,mA}\qquad\begin{matrix}\pi_2\\ nA,\Lambda\vdash\Theta\end{matrix}}{\Gamma,\Lambda\vdash\Delta,\Theta}\bowtie \quad\longrightarrow\quad \bowtie\dfrac{\begin{matrix}\pi_1\\ \Gamma\vdash\Delta,(m+1)A\end{matrix}\qquad\begin{matrix}\pi_2\\ nA,\Lambda\vdash\Theta\end{matrix}}{\Gamma,\Lambda\vdash\Delta,\Theta}$$

since by hp $c(\Pi_1), c(\Pi_2) \leq |A|$ and $c(\pi_i) \leq c(\Pi_i)$ $i=1,2$;
apply the i.h.

Thm TAIT'68 provides an upper bound on the size of the resulting cut-free proof.

Given a proof $\begin{array}{c}\Pi\\ \Gamma\vdash\Delta\end{array}$ with cut-rank $c(\pi)$ there is a cut-free proof $\begin{array}{c}\Pi^*\\ \Gamma\vdash\Delta\end{array}$ such that

$$\ell(\Pi^*) \le \exp(4, c(\pi), \ell(\pi))$$

Thm GENTZEN'35 provide a transformation (algorithm) for LK and LJ (intuitionistic logic in the sequent calculus).