- From the theory of cut-free sequent system some languages (a.l.p.l.) have been derived and produced:

  λ-Prolog
  LOLLI (lin. logic refinement of λ-Prolog) (Miller, Hodas)

provide
- Abstraction
(e.g. Modules, A.D.T, h.o. programming)

But not primitives for concurrency

LO (Linear Objects) (Andreoli & Pareschi)

provides
Primitive for concurrency
but no abstraction.

FORUM

λ-Prolog Lolli LO

- complete pres. of L.L
- ALPL (unif proofs, and log progr style — proof search)
- Abstraction + Concurrency

- Presentation of full linear logic, but considers only half of the connectives: those missing are directly defined as

$$B^\perp \equiv B \multimap \perp \qquad !B \equiv (B \Rightarrow \perp) \multimap \perp$$
$$0 \equiv \top \multimap \perp \qquad B \oplus C \equiv (B^\perp \& C^\perp)^\perp$$
$$1 \equiv \perp \multimap \perp \qquad B \otimes C \equiv (B^\perp \parr C^\perp)^\perp$$
$$\exists x . B \equiv (\forall x . B^\perp)^\perp$$

- The collection of connectives in Forum is not minimal: ? and $\parr$ are primitive in Forum but they could be defined also as
$$?B \equiv (B \multimap \perp) \Rightarrow \perp$$
$$B \parr C \equiv (B \multimap \perp) \multimap C$$

- Forum looks strange:
  - it doesn't use the dualities that follow from negation
  - it mixes both add/mult. connectives
  - it is not minimal

BUT (meta-level) it allows for UNIFORM PROOFS.

- Restricts on the shape of sequents, allowed connectives, inference rules (wrt full LL). Formula building is left free.

Forum sequents and connectives

$$\left[\begin{matrix}\Psi\\ \Gamma\end{matrix}\right] \vdash \left[\begin{matrix}\Xi\\ \Lambda\end{matrix}\right] \qquad \left[\begin{matrix}\Psi\\ \Gamma\end{matrix}\right] A \vdash \left[\Lambda\right]$$

Ψ finite m.set of formulae (classical ctx)

Γ " " " (linear left ctx)

Λ " " atoms

Ξ " sequence formulae

$\top$, ~~ ~~ $\bot$, $⅋$, $\&$, $\Rightarrow$, $\multimap$, $\forall$

$1 \equiv \bot^{\bot}$ $\qquad$ $0 \equiv \top^{\bot}$

$F \otimes F' \equiv (F^{\bot} ⅋ F'^{\bot})^{\bot}$ $\qquad$ $F \oplus F' \equiv (F^{\bot} \& F'^{\bot})^{\bot}$

$!F \equiv (F \Rightarrow \bot)^{\bot}$ $\qquad$ $?F \equiv F^{\bot} \Rightarrow \bot$

$\exists x.F \equiv (\forall x. F^{\bot})^{\bot}$

$F^{\bot} \equiv F \multimap \bot$

Structural Rules

$$\mathsf{i}\;\frac{}{\begin{bmatrix}\Psi\\ \ \end{bmatrix} a \vdash \begin{bmatrix}\ \\ a\end{bmatrix}} \qquad \mathsf{d_L}\;\frac{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} A \vdash \begin{bmatrix}\ \\ \Lambda\end{bmatrix}}{\begin{bmatrix}\Psi\\ \Gamma, A\end{bmatrix} \vdash \begin{bmatrix}\ \\ \Lambda\end{bmatrix}} \qquad \mathsf{d_C}\;\frac{\begin{bmatrix}\Psi, A\\ \Gamma\end{bmatrix} A \vdash \begin{bmatrix}\ \\ \Lambda\end{bmatrix}}{\begin{bmatrix}\Psi, A\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}\ \\ \Lambda\end{bmatrix}} \qquad \mathsf{a}\;\frac{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}\Xi\\ a, \Lambda\end{bmatrix}}{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}\Xi, a\\ \Lambda\end{bmatrix}}$$

Left Rules

$$\bot_{\mathsf{L}}\;\frac{}{\begin{bmatrix}\Psi\\ \ \end{bmatrix} \bot \vdash \begin{bmatrix}\ \\ \ \end{bmatrix}}$$

$$\parr_{\mathsf{L}}\;\frac{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} A \vdash \begin{bmatrix}\ \\ \Lambda\end{bmatrix} \qquad \begin{bmatrix}\Psi\\ \Gamma'\end{bmatrix} B \vdash \begin{bmatrix}\ \\ \Lambda'\end{bmatrix}}{\begin{bmatrix}\Psi\\ \Gamma, \Gamma'\end{bmatrix} A \parr B \vdash \begin{bmatrix}\ \\ \Lambda, \Lambda'\end{bmatrix}}$$

$$\&_{\mathsf{LL}}\;\frac{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} A \vdash \begin{bmatrix}\ \\ \Lambda\end{bmatrix}}{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} A \mathbin{\&} B \vdash \begin{bmatrix}\ \\ \Lambda\end{bmatrix}} \qquad \&_{\mathsf{LR}}\;\frac{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} B \vdash \begin{bmatrix}\ \\ \Lambda\end{bmatrix}}{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} A \mathbin{\&} B \vdash \begin{bmatrix}\ \\ \Lambda\end{bmatrix}}$$

$$\multimap_{\mathsf{L}}\;\frac{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}A\\ \Lambda\end{bmatrix} \qquad \begin{bmatrix}\Psi\\ \Gamma'\end{bmatrix} B \vdash \begin{bmatrix}\ \\ \Lambda'\end{bmatrix}}{\begin{bmatrix}\Psi\\ \Gamma, \Gamma'\end{bmatrix} A \multimap B \vdash \begin{bmatrix}\ \\ \Lambda, \Lambda'\end{bmatrix}}$$

$$\Rightarrow_{\mathsf{L}}\;\frac{\begin{bmatrix}\Psi\\ \ \end{bmatrix} \vdash \begin{bmatrix}A\\ \ \end{bmatrix} \qquad \begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} B \vdash \begin{bmatrix}\ \\ \Lambda\end{bmatrix}}{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} A \Rightarrow B \vdash \begin{bmatrix}\ \\ \Lambda\end{bmatrix}}$$

$$\forall_{\mathsf{L}}\;\frac{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} A[t/x] \vdash \begin{bmatrix}\ \\ \Lambda\end{bmatrix}}{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} \forall x.A \vdash \begin{bmatrix}\ \\ \Lambda\end{bmatrix}}$$

Right Rules

$$\bot_{\mathsf{R}}\;\frac{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}\Xi\\ \Lambda\end{bmatrix}}{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}\bot, \Xi\\ \Lambda\end{bmatrix}}$$

$$\top_{\mathsf{R}}\;\frac{}{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}\top, \Xi\\ \Lambda\end{bmatrix}}$$

$$\parr_{\mathsf{R}}\;\frac{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}A, B, \Xi\\ \Lambda\end{bmatrix}}{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}A \parr B, \Xi\\ \Lambda\end{bmatrix}}$$

$$\&_{\mathsf{R}}\;\frac{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}A, \Xi\\ \Lambda\end{bmatrix} \qquad \begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}B, \Xi\\ \Lambda\end{bmatrix}}{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}A \mathbin{\&} B, \Xi\\ \Lambda\end{bmatrix}}$$

$$\multimap_{\mathsf{R}}\;\frac{\begin{bmatrix}\Psi\\ \Gamma, A\end{bmatrix} \vdash \begin{bmatrix}B, \Xi\\ \Lambda\end{bmatrix}}{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}A \multimap B, \Xi\\ \Lambda\end{bmatrix}}$$

$$\Rightarrow_{\mathsf{R}}\;\frac{\begin{bmatrix}\Psi, A\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}B, \Xi\\ \Lambda\end{bmatrix}}{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}A \Rightarrow B, \Xi\\ \Lambda\end{bmatrix}}$$

$$\forall_{\mathsf{R}}\;\frac{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}A[y/x], \Xi\\ \Lambda\end{bmatrix}}{\begin{bmatrix}\Psi\\ \Gamma\end{bmatrix} \vdash \begin{bmatrix}\forall x.A, \Xi\\ \Lambda\end{bmatrix}}$$

where y is not free in the conclusion

FORUM

Multiset rewriting:

$$\{a, b\}_+ \rightsquigarrow \{c, d, e\}_+$$

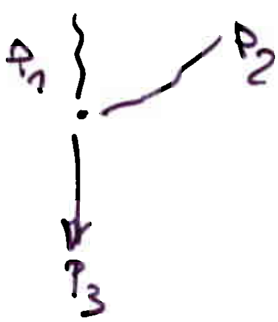
$$a, b \multimap c, d, e$$

if the rule is instead referring to some 'property' (classical reading) you write it with $\Rightarrow$.

Parallel processes:

$$A \parr B$$

Synchronisation:



$$P_1 \parr P_2 \multimap P_3$$

Planning: transformation from an initial to a final state, having available some actions:

$$\mathcal{P} \vdash \mathcal{I} \multimap \mathcal{F}$$

$\mathcal{I}, \mathcal{F}$ are multisets

$\mathcal{P}$ is a multiset of goals.