LINEAR LOGIC : MOTIVATION

- remind the additive, multiplicative versions of cut (context merging / concatenation).
- Similar treatment of ctx, for example, for logical rules ($\wedge_R$)

$$\wedge_R \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \qquad\qquad \wedge_R \frac{\Gamma \vdash \Delta, A \qquad \Lambda \vdash \Theta, B}{\Gamma, \Lambda \vdash \Delta, \Theta, A \wedge B}$$

(additive) (multiplicative)

- You can simulated the 1st by means of the 2nd + structural rules (weakening and contr.) and viceversa : so they are equivalent.

QUESTION What happens if weakening and contraction are not available ??

THE TWO FORMS ARE NOT EQUIVALENT ANYMORE

⟶ RE INTRODUCE CONTR + WEAK. IN A CONTROLLED FORM TO GET CLASSICAL LOGIC

THIS IS THE BASIC IDEA OF LINEAR LOGIC.

LINEAR LOGIC : CONNECTIVES

	mult.	add
$\wedge$	$\otimes$	$\&$
$\vee$	$⅋$	$\oplus$

(tensor, with)

(par, plus)

+ modalities : ! ("bang" or "of-course")

? ("why-not")

$A^\perp$: negation, is involutive

- A more structured presentation makes use of one-sided sequents.

Conjunction becomes

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \otimes_L \qquad \otimes_R \frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}$$

(tensor product, or cumulative conjunction)

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \&_{LL} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \&_{LR}$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \& B, \Delta} \&_R$$

(direct product or alternative conj.)

~~Disjunction~~ becomes:

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \oplus_L \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus_{RL}$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus_{RR}$$

(direct sum, alternative disj.)

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A ⅋ B \vdash \Delta, \Delta'} ⅋_L \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A ⅋ B, \Delta} ⅋_R$$

(tensor sum, cumulative disj.).

Negation

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, A^\perp \vdash \Delta} \perp_L \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^\perp, \Delta} \perp_R$$

- give atomic formulas in two forms:

$$A \qquad A^\perp$$

Define the negation of A as $A^\perp$ and the negation $A^{\perp\perp}$ of $A^\perp$ is A.

- Define using De Morgan:

$$(A \otimes B)^\perp = A^\perp \parr B^\perp$$
$$(A \parr B)^\perp = A^\perp \otimes B^\perp$$

$$(A \,\&\, B)^\perp = A^\perp \oplus B^\perp$$
$$(A \oplus B)^\perp = A^\perp \,\&\, B^\perp$$

- Define linear implication:

$A \multimap B$ stands for $A^\perp \parr B$

- Convert a symmetric sequent

$A_1 \ldots A_n \vdash B_1, \ldots B_m$ into

$$\vdash A_1^\perp \ldots A_n^\perp, B_1 \ldots B_m$$

- identity axiom: $\vdash A^\perp, A$

this subsumes the rules for negation (lin. neg.)

- Cut becomes:

$$\frac{\vdash A, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\text{ Cut}$$

- ~~Exchange~~ rule:

$$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\, x$$

- with these things the logical rules become:

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\, \parr \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\, \otimes$$

$$\frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\, \oplus_1 \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\, \oplus_2 \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \& B, \Gamma}\, \&$$

- FULL SYSTEM for PROP. LIN. LOG.: introduce units!

$$1^\perp = \perp \qquad \perp^\perp = 1 \qquad \top^\perp = 0 \qquad 0^\perp = \top$$

$$\frac{\vdash \Gamma}{\vdash \perp, \Gamma}\, \perp \qquad \frac{}{\vdash 1}\, 1 \qquad \text{(no rule for 0)} \qquad \frac{}{\vdash \top, \Gamma}\, \top$$

1	$\perp$	$\top$	0	are units for
$\otimes$	$\parr$	$\&$	$\oplus$	

- Modalities: $(!A)^\perp = ?A^\perp$ $\quad (?A)^\perp = !A^\perp$

$$\frac{\vdash \Gamma}{\vdash ?A, \Gamma}\, w? \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\, c? \qquad \frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\, D? \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\, !$$

(weakening) (contraction) dereliction promotion

- ~~Some equivalences~~:

$$!(A \& B) \equiv (!A) \otimes (!B)$$

$$?(A \oplus B) \equiv (?A) ⅋ (?B)$$

- Remark: the rules for ! reminds the rules for modal connectives $\Box$ and $\Diamond$.

- Remark: 'Commas' are multiplicative

$$\Gamma \vdash \Delta \quad \text{means}$$

$$\otimes \Gamma \vdash ⅋ \Delta$$

- N.B.: Unprovable formulas in LL: (check! These are provable in C.L. if you substitute $\wedge$ to $\otimes$)

$$\vdash A \otimes A, \; A^\perp \otimes A^\perp$$

$$\vdash A \otimes B, \; A^\perp \otimes B, \; A \otimes B^\perp, \; A^\perp \otimes B^\perp$$

$$\vdash (A \otimes A) \multimap A$$

$$\vdash A \multimap (A \otimes A)$$

$\rightsquigarrow$ $\vdash A^\perp \oplus A$ That's why it doesn't make sense to define an additive implication.

Informal semantics of linear logic connectives

FIXED PRICE MENU FOR 20 EUROS

Soup: Fish Soup — FS
Vegetable Soup — VS

Main: Salmon — S
Mutton — M
Cheese — C

Dessert or Fruit:
(Banana, Apple, Grape)* — Ba Ap, Gr
(Ice-cream, Cake, Surprise of the day*) — Ice, Cak Sur

*(depending on availability)

$$20EUR^{\perp} \parr$$
$$(FS \,\&\, VS) \otimes$$
$$(S \,\&\, M \,\&\, C) \otimes$$
$$((Ba \oplus Ap \oplus Gr) \,\&\, (Ice \oplus Cak \oplus Sur))$$

$A^{\perp} \parr B$ is the def of $A \multimap B$
(multiset rewriting $\{A\}_+ \rightsquigarrow \{B\}_+$)
'consume A to get a B')

$A \,\&\, B$: get just one between A or B

$A \otimes B$ get both

$A \oplus B$ get one but you can't decide

~~TERMINOLOGY~~

- Many fragments in linear logic have a specific name.
- Studies in complexity theory

LL	full linear logic
MLL	multiplicative L.L. (no exponentials)
MELL	multiplicative exponential L.L.
MALL	multiplicative additive L.L. (no exp.)

- LL enjoys cut-elimination and the subformula property

- We can represent intuitionistic logic into LL and also classical logic

Idea
1) Define a translation for formulae on IL into LL
2) Prove correctness (of the translation) and completeness

Ex: I.L. is translated as

$$(A \supset B)^* = ?(A^*)^\perp \parr B = (!A^*) \multimap B^*$$
$$(A \wedge B)^* = A^* \& B^*$$
$$(A \vee B)^* = !A^* \oplus !B^*$$